COMPUTATIONAL PDE LECTURE 16

LUCAS BOUCK

1. Outline of today

- Prove ∞-norm estimates for finite difference methods for the heat equation
- Extend analysis to Crank-Nicolson

2. Finite Difference Methods for the Heat Equation

We are trying to solve:

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = f(t,x), & t \in (0,1), x \in (0,1) \\ u(t,0) = 0, u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

We now list with various time stepping schemes for the heat equation, which you implemented in recitation:

- **Forward Euler** (approximates differential equation at t_{j-1})

$$D_\tau \mathbf{U}^j + \mathbf{A}^h \mathbf{U}^{j-1} = \mathbf{f}^{j-1}$$

- **Backward Euler** (approximates differential equation at t_j)

$$D_\tau \mathbf{U}^j + \mathbf{A}^h \mathbf{U}^j = \mathbf{f}^j$$

2.1. Error estimate for Backward Euler.

Recall that we have shown consistency of both schemes.

Proposition 2.1 (consistency of schemes)**.** Let u solve (1). Then $\mathbf{u}_i^j = u(t_j, x_i)$ satisfies

- **Forward Euler**

$$D_\tau \mathbf{u}^j + \mathbf{A}^h \mathbf{u}^{j-1} = \mathbf{f}^{j-1} + \boldsymbol{\tau}^j_{\tau,h,fe}$$

- **Backward Euler**

$$D_\tau \mathbf{u}^j + \mathbf{A}^h \mathbf{u}^j = \mathbf{f}^j + \boldsymbol{\tau}^j_{\tau,h,be}$$

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where there is a constant $C > 0$ such that the truncation errors satisfy:

$$\|\boldsymbol{\tau}^j_{\tau,h,fe}\|_\infty \le C\left(\tau|u_{tt}|_{max} + h^2|u_{xxxx}|_{max}\right)$$
$$\|\boldsymbol{\tau}^j_{\tau,h,be}\|_\infty \le C\left(\tau|u_{tt}|_{max} + h^2|u_{xxxx}|_{max}\right)$$

and

$$|u_{tt}|_{max} = \max_{t,x\in[0,1]} |u_{tt}(t,x)|, \quad |u_{ttt}|_{max} = \max_{t,x\in[0,1]} |u_{ttt}(t,x)|, \quad |u_{xxxx}|_{max} = \max_{t,x\in[0,1]} |u_{xxxx}(t,x)|$$

Recall that we have

Proposition 2.2 (∞-norm stability of Euler)**.** Let $\mathbf{U}^j$ be the sequence of solutions to the Backward Euler scheme. Then the vector $\mathbf{U}^J$ satisfies:

$$\|\mathbf{U}^J\|_\infty \le \|\mathbf{U}^0\|_\infty + \sum_{j=1}^{J} \tau\|\mathbf{f}^j\|_\infty.$$

Let $\mathbf{U}^j$ be the sequence of solutions to the Backward Euler scheme. Then if

$$\tau \le \frac{h^2}{2} \tag{2}$$

the vector $\mathbf{U}^j$ satisfies:

$$\|\mathbf{U}^J\|_\infty \le \|\mathbf{U}^0\|_\infty + \sum_{j=0}^{J-1} \tau\|\mathbf{f}^j\|_\infty.$$

The above stability result leads to the following error estimate:

Proposition 2.3 (∞-norm error estimate of Euler)**.** Let $\mathbf{U}^j$ be the sequence of solutions to either the Forward or Backward Euler scheme. Then for any h, τ for B.E. or $\tau \le \frac{h^2}{2}$ for F.E., the error vector $\mathbf{e}^J = \mathbf{u}^J - \mathbf{U}^J$ satisfies:

$$\|\mathbf{e}^J\|_\infty \le CT\left(\tau|u_{tt}|_{max} + h^2|u_{xxxx}|_{max}\right).$$

Proof. We have that $\mathbf{e}^j = \mathbf{u}^j - \mathbf{U}^j$ solves

$$D_\tau \mathbf{e}^j + \mathbf{A}^h\mathbf{e}^j = \boldsymbol{\tau}^j_{\tau,h}$$

and apply stability result with estimates on truncation error. □

Remark 2.1 (stability influence on error estimates)**.** The kind of stability we prove determines what kind of error estimate we expect. The following table shows the relevant PDE stability, discrete stability, and error estimate.

PDE Stability	Energy Estimates	Max Principle
Discrete Analog	$\|\mathbf{U}^J\|^2_{2,h} \le \|\mathbf{U}^0\|^2_{2,h} + \sum_{j=1}^J \tau\|\mathbf{f}^j\|_{2,h}$	$\|\mathbf{U}^J\|_\infty \le \|\mathbf{U}^0\|_\infty + \sum_{j=1}^J \tau\|\mathbf{f}^j\|_{2,h}$
Error Estimate	$\|\mathbf{e}^J\|_{2,h} \le \mathcal{O}(\sqrt{T}(h^2+\tau))$	$\|\mathbf{e}^J\|_\infty \le \mathcal{O}(T(h^2+\tau))$

3. Extending to Crank-Nicolson

Recall the Crank Nicolson iteration is an approximation of the PDE at $t_{j-1/2}$:

$$D_\tau \mathbf{U}^j + \mathbf{A}^h \frac{\mathbf{U}^j + \mathbf{U}^{j-1}}{2} = \mathbf{f}^{j-1/2}$$

and we had the following truncation error result.

Proposition 3.1 (consistency of CN)**.** Let u solve (1). Then $\mathbf{u}_i^j = u(t_j, x_i)$ satisfies

$$D_\tau \mathbf{u}^j + \frac{1}{2}\mathbf{A}^h(\mathbf{u}^j + \mathbf{u}^{j-1}) = \mathbf{f}^{j+1/2} + \boldsymbol{\tau}^j_{\tau,h,cn}$$

where there is a constant $C > 0$ such that the truncation error satisfies:

$$\|\boldsymbol{\tau}^j_{\tau,h,cn}\|_\infty \leq C\left(\tau^2|u_{ttt}|_{max} + \tau^2|u_{xxtt}|_{max} + h^2|u_{xxxx}|_{max}\right)$$

If we think of the relevant stability being energy estimates, then the discrete analog for Crank-Nicolson is

Proposition 3.2 (energy estimates of CN)**.** Let $\mathbf{U}^j$ solve the iteration:

$$D_\tau \mathbf{U}^j + \mathbf{A}^h \frac{\mathbf{U}^j + \mathbf{U}^{j-1}}{2} = \mathbf{f}^{j-1/2}.$$

Then,

$$\|\mathbf{U}^J\|_{2,h} \leq \|\mathbf{U}^0\|_{2,h} + \sum_{j=1}^{J} \tau \|\mathbf{f}^{j-1/2}\|_{2,h}.$$

Proof. Will be a HW exercise. Multiply equation with $\frac{\mathbf{U}^j + \mathbf{U}^{j-1}}{2}$. □

Once we have a stability estimate, we can then immediately know the correct error estimate.

Proposition 3.3 (error estimates of CN)**.** Let $\mathbf{U}^j$ solve the CN iteration: Then the error $\mathbf{e}^j = \mathbf{u}^j - \mathbf{U}^j$ satisfies

$$\|\mathbf{e}^J\|_{2,h} \leq TC\left(\tau^2|u_{ttt}|_{max} + \tau^2|u_{xxtt}|_{max} + h^2|u_{xxxx}|_{max}\right).$$

Proof. Write down CN iteration for $\mathbf{e}^j$ and apply stability with estimate on truncation error. □