COMPUTATIONAL PDE LECTURE 14

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1. Outline of today

- Start finite differences for the heat equation.

2. Finite Difference Methods for the Heat Equation

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = f(t,x), \quad t \in (0,1), x \in (0,1) \\ u(t,0) = 0, u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

2.1. **Time stepping schemes.** We now list with various time stepping schemes for the heat equation, which you implemented in recitation:

- **Forward Euler** (approximates differential equation at t_{j-1})

$$D_\tau \mathbf{U}^j + \mathbf{A}^h \mathbf{U}^{j-1} = \mathbf{f}^{j-1}$$

- **Backward Euler** (approximates differential equation at t_j)

$$D_\tau \mathbf{U}^j + \mathbf{A}^h \mathbf{U}^j = \mathbf{f}^j$$

- **Crank Nicolson** (approximates differential equation at $t_{j-\frac{1}{2}} = \frac{1}{2}(t_j + t_{j-1})$)

$$D_\tau \mathbf{U}^j + \frac{1}{2}\mathbf{A}^h(\mathbf{U}^j + \mathbf{U}^{j-1}) = \mathbf{f}^{j-1/2}$$

2.2. **Stability: discrete energy estimates.** The other key ingredient is stability. We'll mimic the energy estimates from the continuous heat equation. We first start with Backward Euler:

Proposition 2.1 (unconditional stability of Backward Euler)**.** Let $\mathbf{U}^j$ be the sequence of solutions to the Backward Euler scheme:

$$D_\tau \mathbf{U}^j + \mathbf{A}^h \mathbf{U}^j = \mathbf{f}^j,$$

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then the vector $\mathbf{U}^j$ satisfies the following discrete energy estimate:

$$\|\mathbf{U}^J\|_{2,h}^2 + \sum_{j=1}^{J}\left[\tau\|\mathbf{U}^J\|_{\mathbf{A}^h}^2 + \|\mathbf{U}^j - \mathbf{U}^{j-1}\|_{2,h}^2\right] \le \|\mathbf{U}^0\|_{2,h} + \sum_{j=1}^{J}\tau\|\mathbf{f}^j\|_{2,h}^2 \tag{2}$$

where

$$\|\mathbf{v}\|_{2,h}^2 = \sum_{i=1}^{N-1} h\mathbf{v}_i^2, \quad \|\mathbf{v}\|_{\mathbf{A}^h}^2 = h\mathbf{v}^T\mathbf{A}^h\mathbf{v}.$$

Remark 2.1 (discrete norms)**.** Note that we have the following Riemann sum approximation:

$$\|\mathbf{u}^j\|_{2,h}^2 = \sum_{i=1}^{N-1} hu(t_j, x_i)^2 \approx \int_0^1 u(t_j, x)^2 dx.$$

I claim that the $\mathbf{A}^h$ norm for a vector is a Riemann sum approximation of the square integral of a derivative

$$\|\mathbf{u}^j\|_{\mathbf{A}^h}^2 \approx \int_0^1 |u_x(t_j, x)|^2 dx.$$

To see this, we compute

$$\|\mathbf{u}^j\|_{\mathbf{A}^h}^2 = \sum_{i=1}^{N-1} hu(t_j, x_i)\left(\frac{-u(t_j, x_{i-1}) + 2u(t_j, x_i) - u(t_j, x_{i+1})}{h^2}\right)$$

We observe that this is a Riemann sum approximation for:

$$\|\mathbf{u}^j\|_{\mathbf{A}^h}^2 \approx -\int_0^1 u(t_j, x)u_{xx}(t_j, x)dx,$$

and we integrate by parts to get:

$$\|\mathbf{u}^j\|_{\mathbf{A}^h}^2 \approx -\int_0^1 u(t_j, x)u_{xx}(t_j, x)dx = \int_0^1 |u_x(t_j, x)|^2 dx$$

As a result, the estimate (2) can be viewed as a discrete analog to the energy estimates we have proved in earlier lectures:

$$\int_0^1 u(T, x)^2 dx + \int_0^T \int_0^1 u_x(t, x)^2 dx \le \int_0^1 u(0, x)^2 dx + \int_0^T \int_0^1 f(t, x)^2 dx$$

The additional term:

$$\sum_{j=1}^{J}\left[\|\mathbf{U}^j - \mathbf{U}^{j-1}\|_{2,h}^2\right]$$

is known as **numerical dissipation**. This term shows that the numerical method dissipates more energy than expected from the PDE. This is ok because

$$\mathbf{U}^j - \mathbf{U}^{j-1} = \tau D_\tau \mathbf{U}^j = \mathcal{O}(\tau),$$

and

$$\sum_{j=1}^{J} \left[\|\mathbf{U}^j - \mathbf{U}^{j-1}\|_{2,h}^2 \right] = \mathcal{O}(\tau),$$

so this extra dissipation is small.

Proof of Discrete Energy Estimate. Just to mimic the continuous case, we take a dot product of the discrete evolution equation with $h\mathbf{U}^j$ to get

$$h\mathbf{U}^j \cdot D_\tau \mathbf{U}^j + \|\mathbf{U}^j\|_{\mathbf{A}^h}^2 = h\mathbf{U}^j \cdot \mathbf{f}^j$$

We write out the first term

$$h\mathbf{U}^j \cdot D_\tau \mathbf{U}^j = \frac{h}{\tau} \mathbf{U}^j \cdot \left(\mathbf{U}^j - \mathbf{U}^{j-1} \right),$$

and use the following fact for vectors:

$$\mathbf{a} \cdot (\mathbf{a} - \mathbf{b}) = \frac{1}{2}\|\mathbf{a}\|_2^2 - \frac{1}{2}\|\mathbf{b}\|_2^2 + \frac{1}{2}\|\mathbf{a} - \mathbf{b}\|_2^2.$$

Hence, with $\mathbf{a} = \mathbf{U}^j$, and $\mathbf{b} = \mathbf{U}^{j-1}$, we have

$$\begin{aligned} h\mathbf{U}^j \cdot D_\tau \mathbf{U}^j &= \frac{h}{\tau} \left(\frac{1}{2}\|\mathbf{U}^j\|_2^2 - \frac{1}{2}\|\mathbf{U}^{j-1}\|_2^2 + \frac{1}{2}\|\mathbf{U}^j - \mathbf{U}^{j-1}\|_2^2 \right) \\ &= \frac{1}{\tau} \left(\frac{1}{2}\|\mathbf{U}^j\|_{2,h}^2 - \frac{1}{2}\|\mathbf{U}^{j-1}\|_{2,h}^2 + \frac{1}{2}\|\mathbf{U}^j - \mathbf{U}^{j-1}\|_{2,h}^2 \right). \end{aligned}$$

The discrete equation now reads

$$\frac{1}{\tau} \left(\frac{1}{2}\|\mathbf{U}^j\|_{2,h}^2 - \frac{1}{2}\|\mathbf{U}^{j-1}\|_{2,h}^2 + \frac{1}{2}\|\mathbf{U}^j - \mathbf{U}^{j-1}\|_{2,h}^2 \right) + \|\mathbf{U}^j\|_{\mathbf{A}^h}^2 = h\mathbf{U}^j \cdot \mathbf{f}^j$$

We estimate the RHS using Young's inequality

$$h\mathbf{U}^j \cdot \mathbf{f}^j = h \sum_{i=1}^{N-1} \mathbf{U}_i^j \mathbf{f}_i^j \leq \frac{h}{2} \sum_{i=1}^{N-1} (\mathbf{U}_i^j)^2 + (\mathbf{f}_i^j)^2 = \frac{1}{2} \left(\|\mathbf{U}^j\|_{2,h}^2 + \|\mathbf{f}^j\|_{2,h}^2 \right)$$

Just like in the proof of energy estimates for the heat equation, we need a Poincare inequality:

$$\|\mathbf{U}^j\|_{2,h}^2 \leq \|\mathbf{U}^j\|_{\mathbf{A}^h}^2,$$

which we will prove in the next Lemma. Once we have the above inequality, the discrete equation is now an inequality:

$$\frac{1}{\tau}\left(\frac{1}{2}\|\mathbf{U}^j\|_{2,h}^2 - \frac{1}{2}\|\mathbf{U}^{j-1}\|_{2,h}^2 + \frac{1}{2}\|\mathbf{U}^j - \mathbf{U}^{j-1}\|_{2,h}^2\right) + \|\mathbf{U}^j\|_{\mathbf{A}^h}^2 \le \frac{1}{2}\|\mathbf{U}^j\|_{\mathbf{A}^h}^2 + \frac{1}{2}\|\mathbf{f}^j\|_{2,h}^2$$

Subtracting $\frac{1}{2}\|\mathbf{U}^j\|_{\mathbf{A}^h}^2$ from both sides and multiplying both sides by 2 leads to

$$\frac{1}{\tau}\left(\|\mathbf{U}^j\|_{2,h}^2 - \|\mathbf{U}^{j-1}\|_{2,h}^2 + \|\mathbf{U}^j - \mathbf{U}^{j-1}\|_{2,h}^2\right) + \|\mathbf{U}^j\|_{\mathbf{A}^h}^2 \le \|\mathbf{f}^j\|_{2,h}^2$$

Finally, multiplying by τ and summing from $j = 1, \ldots, J$ yields

$$\sum_{j=1}^{J}\left(\|\mathbf{U}^j\|_{2,h}^2 - \|\mathbf{U}^{j-1}\|_{2,h}^2\right) + \sum_{j=1}^{J}\tau\|\mathbf{U}^j\|_{\mathbf{A}^h}^2 + \|\mathbf{U}^j - \mathbf{U}^{j-1}\|_{2,h}^2 \le \sum_{j=1}^{J}\tau\|\mathbf{f}^j\|_{2,h}^2$$

Notice the sum on the LHS telescopes:

$$\sum_{j=1}^{J}\left(\|\mathbf{U}^j\|_{2,h}^2 - \|\mathbf{U}^{j-1}\|_{2,h}^2\right) = \|\mathbf{U}^J\|_{2,h}^2 - \|\mathbf{U}^0\|_{2,h}^2,$$

which gives the result. □

The important Poincare inequality lemma we need is a property of the matrix $\mathbf{A}^h$.

Lemma 2.1 (eigenvalues of $\mathbf{A}^h$)**.** The eigenvalues of $\mathbf{A}^h \in \mathbb{R}^{(N-1)\times(N-1)}$ are

$$\lambda_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right), \quad k = 1, \ldots, N-1$$

with eigenvectors

$$\mathbf{v}_i^k = \sin(k\pi x_i).$$

Proof. Left as exercise to verify that

$$\frac{1}{h^2}\left(-\mathbf{v}_{i-1}^k + 2\mathbf{v}_i^k - \mathbf{v}_{i+1}^k\right) = \lambda_k \mathbf{v}_i^k$$

□

A consequence of this property is that we have a discrete Poincare inequality:

Lemma 2.2 (discrete Poincare inequality)**.** For any vector $\mathbf{v}$, we have

$$\|\mathbf{v}\|_{2,h}^2 \le \lambda_1\|\mathbf{v}\|_{2,h}^2 \le \|\mathbf{v}\|_{\mathbf{A}^h}^2 \le \lambda_{N-1}\|\mathbf{v}\|_{2,h}^2 \le \frac{4}{h^2}\|\mathbf{v}\|_{2,h}^2.$$

In particular, we have

$$\|\mathbf{v}\|_{2,h}^2 \le \|\mathbf{v}\|_{\mathbf{A}^h}^2$$

Proof. The inequality $\lambda_1 \geq 1$ is not entirely obvious, so we prove it. Note that

$$\lambda_1 = \left(\frac{2}{h} \sin\left(\frac{\pi h}{2} \right) \right)^2,$$

so it is sufficient to prove that $\frac{2}{h} \sin\left(\frac{\pi h}{2}\right) \geq 1$. This is true, but we only sketch the idea. As $h \to 0$, we have that $h_2 = h/2 \to 0$ and by the limit definition of derivative:

$$\lim_{h \to 0} \frac{2}{h} \sin\left(\frac{\pi h}{2} \right) = \lim_{h \to 0} \frac{\sin(\pi h_2)}{h_2} = \pi \cos(0) = \pi.$$

Hence, at least for sufficiently small h, we expect $\lambda_1 \geq 1$. $\square$