COMPUTATIONAL PDE LECTURE 13

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1. Outline of today

- Start finite differences for the heat equation.

2. Finite Difference Methods for the Heat Equation

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = f(t,x), \quad t \in (0,1), x \in (0,1) \\ u(t,0) = 0, u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

We now begin the discussion of finite difference methods for the heat equation. We begin with the setup:

- $x_j = jh$ spatial grid points $h = 1/N$
- $t_j = j\tau$ time grid points $\tau = T/M$
- Grid $G_{\tau,h} = \{t_j\}_{j=0}^M \times \{x_i\}_{i=0}^N$
- Grid function $U^{h,\tau} : G_{\tau,h} \to \mathbb{R}$

To highlight the role of time stepping, we denote $\mathbf{U}^j \in \mathbb{R}^{N-1}$ as the vector of $U^{h,\tau}$ evaluated at interior grid points, i.e.

$$\mathbf{U}_i^j = U^{h,\tau}(t_j, x_i) \text{ for } i = 1, \dots, N-1$$

and the negative second finite difference for $U^{h,\tau}(t_j, x_i)$ will be denoted by

$$\mathbf{A}^h = \begin{pmatrix} \frac{2}{h^2} & \frac{-1}{h^2} & & \\ -\frac{1}{h^2} & \frac{2}{h^2} & \frac{-1}{h^2} & \\ & \ddots & \ddots & \ddots \end{pmatrix},$$

so that

$$(\mathbf{A}^h \mathbf{U}^j)_i = -D_h^2 U^{h,\tau}(t_j, x_i).$$

We finally denote the a finite difference in time as

$$D_\tau U^{h,\tau}(t_j, x_i) = \frac{U^{h,\tau}(t_j, x_i) - U^{h,\tau}(t_{j-1}, x_i)}{\tau}$$

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or

$$D_\tau \mathbf{U}^j = \frac{\mathbf{U}^j - \mathbf{U}^{j-1}}{\tau}$$

2.1. **Time stepping schemes.** We now list with various time stepping schemes for the heat equation, which you implemented in recitation:

- **Forward Euler** (approximates differential equation at t_{j-1})

$$D_\tau \mathbf{U}^j + \mathbf{A}^h \mathbf{U}^{j-1} = \mathbf{f}^{j-1}$$

- **Backward Euler** (approximates differential equation at t_j)

$$D_\tau \mathbf{U}^j + \mathbf{A}^h \mathbf{U}^j = \mathbf{f}^j$$

- **Crank Nicolson** (approximates differential equation at $t_{j-\frac{1}{2}} = \frac{1}{2}(t_j + t_{j-1})$)

$$D_\tau \mathbf{U}^j + \frac{1}{2}\mathbf{A}^h(\mathbf{U}^j + \mathbf{U}^{j-1}) = \mathbf{f}^{j-1/2}$$

2.2. **Consistency: truncation Errors.** Recall that there are two ingredients to demonstrating convergence for finite difference schemes, which were:

- Consistency,
- Stability.

Typically it is easier to show consistency using Taylor expansion.

Proposition 2.1 (consistency of schemes)**.** Let u solve (1). Then $\mathbf{u}_i^j = u(t_j, x_i)$ satisfies

- **Forward Euler**

$$D_\tau \mathbf{u}^j + \mathbf{A}^h \mathbf{u}^{j-1} = \mathbf{f}^{j-1} + \boldsymbol{\tau}^j_{\tau,h,fe}$$

- **Backward Euler**

$$D_\tau \mathbf{u}^j + \mathbf{A}^h \mathbf{u}^j = \mathbf{f}^j + \boldsymbol{\tau}^j_{\tau,h,be}$$

- **Crank Nicolson**

$$D_\tau \mathbf{u}^j + \frac{1}{2}\mathbf{A}^h(\mathbf{u}^j + \mathbf{u}^{j-1}) = \mathbf{f}^{j+1/2} + \boldsymbol{\tau}^j_{\tau,h,cn}$$

where there is a constant $C > 0$ such that the truncation errors satisfy:

$$\begin{aligned}
\|\boldsymbol{\tau}^j_{\tau,h,fe}\|_\infty &\leq C\left(\tau|u_{ttt}|_{C^0} + h^2|u_{xxxx}|_{C^0}\right)\\
\|\boldsymbol{\tau}^j_{\tau,h,be}\|_\infty &\leq C\left(\tau|u_{ttt}|_{C^0} + h^2|u_{xxxx}|_{C^0}\right)\\
\|\boldsymbol{\tau}^j_{\tau,h,cn}\|_\infty &\leq C\left(\tau^2|u_{ttt}|_{C^0} + \tau^2|u_{xxtt}|_{C^0} + h^2|u_{xxxx}|_{C^0}\right)
\end{aligned}$$

and

$$|u_{ttt}|_{C^0} = \max_{t,x\in[0,1]} |u_{tt}(t,x)|, \quad |u_{ttt}|_{C^0} = \max_{t,x\in[0,1]} |u_{ttt}(t,x)|, \quad |u_{xxxx}|_{C^0} = \max_{t,x\in[0,1]} |u_{xxxx}(t,x)|$$

Proof. Recall that we have already shown:

$$\left| u_{xx}(t_j, x_i) - \frac{u(t_j, x_{i+1}) - 2u(t_j, x_i) + u(t_j, x_{i-1})}{h^2} \right| \leq Ch^2 |u_{xxxx}|_{C^0}.$$

We have seen in HW2 for a backward difference formula

$$\left| u_t(t_{j-1}, x_i) - \frac{u(t_j, x_i) - u(t_{j-1}, x_i)}{\tau} \right| \leq C\tau |u_{tt}|_{C^0}.$$

Note that we picked to evaluate the time derivative of u at t_{j-1}. For backward Euler, we'd want to look at $u_t(t_j, x_i)$. Thus, from HW2, we have the following truncation errors for each time derivative approximation

$$\textbf{Forward Euler}: \left| u_t(t_{j-1}, x_i) - \frac{u(t_j, x_i) - u(t_{j-1}, x_i)}{\tau} \right| \leq C\tau |u_{tt}|_{C^0}$$

$$\textbf{Backward Euler}: \left| u_t(t_j, x_i) - \frac{u(t_j, x_i) - u(t_{j-1}, x_i)}{\tau} \right| \leq C\tau |u_{tt}|_{C^0}$$

$$\textbf{Crank Nicolson}: \left| u_t((t_j + t_{j-1})/2, x_i) - \frac{u(t_j, x_i) - u(t_{j-1}, x_i)}{\tau} \right| \leq C\tau^2 |u_{ttt}|_{C^0}$$

The proof is complete for Forward Euler and Backward Euler.

Notice that Crank Nicolson is a centered difference approximation of u_t at the midpoint between t_j and t_{j-1}. So far we have shown for CN:

$$\begin{aligned} D_\tau u(t_j, x_i) - \frac{1}{2} D_h^2 \left(u(t_j, x_i) + u(t_{j-1}, x_i) \right) &= u_t(t_{j-1/2}, x_i) - \frac{1}{2} \left(u_{xx}(t_j, x_i) + u_{xx}(t_{j-1}, x_i) \right) + (\tilde{\boldsymbol{\tau}}^j_{h,\tau,cn})_i \\ &= u_t(t_{j-1/2}, x_i) - u_{xx}(t_{j-1/2}, x_i) + a_{i,j} + (\tilde{\boldsymbol{\tau}}^j_{h,\tau,cn})_i \\ &= f(t_{j-1/2}, x_i) + a_{i,j} + (\tilde{\boldsymbol{\tau}}^j_{h,\tau,cn})_i \end{aligned}$$

where

$$\|\tilde{\boldsymbol{\tau}}^j_{h,\tau,cn}\|_\infty \leq C\tau^2 |u_{ttt}|_{C^0} + Ch^2 |u_{xxxx}|_{C^0}$$

and

$$a_{i,j} = u_{xx}(t_{j-1/2}, x_i) - \frac{1}{2} \left(u_{xx}(t_j, x_i) + u_{xx}(t_{j-1}, x_i) \right).$$

To complete the proof of the truncation error for CN, one can show using Taylor expansion that

$$|a_{i,j}| \leq \tau^2 |u_{xxtt}|_{C^0},$$

which will be left as a HW problem. □