COMPUTATIONAL PDE LECTURE 12

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1. Outline of today

- Finish separation of variables
- Start finite differences

2. Neumann boundary conditions

This section address what happens when we have homogenous Neumann boundary conditions:

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = 0, \quad t > 0, x \in (0,1) \\ u_x(t,0) = 0, u_x(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

We again write the solution as

$$u(t,x) = \sum_{k=1}^{\infty} T_k(t)\tilde{X}_k(x)$$

where $\tilde{X}_k$ is a different function than $X_k(x) = \sin(k\pi x)$. This is because X_k does not satisfy the boundary conditions $X_k'(0) = X_k'(1) = 0$. Plugging u into the heat equation yields

$$\sum_{k=1}^{\infty} \Big(T_k'(t)\tilde{X}_k(x) - T_k(t)\tilde{X}_k''(x) \Big) = 0.$$

Solving for each individual k brings us back to

$$\frac{T_k'(t)}{T_k(t)} = \frac{X_k''(x)}{X_k(x)} = \tilde{\lambda}_k$$

where $\tilde{\lambda}_k$ is a constant. Therefore, we need to solve the eigenvalue problem

$$\begin{cases} \tilde{X}''(x) - \tilde{\lambda}\tilde{X}(x) = 0 \\ \tilde{X}'(0) = \tilde{X}'(1) = 0 \end{cases} . \tag{2}$$

We now go over how to solve these eigenvalue problems.

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Proposition 2.1. The values $\tilde{\lambda}_k = -(k\pi)^2$ and $\tilde{X}_k(x) = \cos(k\pi x)$ for $k = 0, \dots$ are the only nonzero solutions to the Neumann eigenvalue problem (2).

Proof. We break the proof into 2 cases.

Case 1: Suppose $\tilde{X}, \tilde{\lambda}$ solve (2) with $\tilde{\lambda} > 0$. Then we have that $\tilde{\lambda} = a^2$ for some $a > 0$. Then the solution to $\tilde{X}'' = a^2\tilde{X}$ is

$$\tilde{X}(x) = c_1 e^{ax} + c_2 e^{-ax}.$$

Notice that $\tilde{X}'(0) = c_1 a - c_2 a$, so in order to satisfy $\tilde{X}'(0) = 0$, we require $c_1 = c_2$. If this is the case, then $\tilde{X}'(0) = c_1 a e^a - c_1 a e^{-a} = c_1 a(e^a - e^{-a})$. The only way for $c_1 a(e^a - e^{-a}) = 0$ for $a > 0$ is if $c_1 = 0$. Thus, $\tilde{X}(x) = 0$. Hence there cannot be a nonzero eigenfunction with $\tilde{\lambda} > 0$.

Case 2: Suppose $\tilde{X}, \tilde{\lambda}$ solve (2) with $\tilde{\lambda} \leq 0$. Then we have that $\tilde{\lambda} = -a^2$ for some $a \geq 0$. The solution to $\tilde{X}'' = -a^2\tilde{X}$ is

$$\tilde{X}(x) = c_1 \cos(ax) + c_2 \sin(ax).$$

To satisfy the boundary conditions, we require

$$\begin{aligned} 0 &= \tilde{X}'(0) = \cancel{-ac_1 \sin(a0)} + ac_2 \cos(a0) = ac_2 \\ 0 &= \tilde{X}'(0) = -ac_1 \sin(a) + ac_2 \cos(a) \end{aligned}$$

To solve the first equation, we require $a = 0$ or $c_2 = 0$.

If $a = 0$, then

$$\tilde{X}(x) = c_1$$

is a constant. Since we do not care about the constant scaling of an eigenvector, we take $c_1 = 1$.

If $a > 0$, then $c_2 =$, and we require $-ac_1 \sin(a) = 0$. In order for $\tilde{X} \neq 0$, we need $\sin(a) = 0$, and the only solutions for $a > 0$ are $a = \pi k$ for $k = 1, \dots$.

In conclusion, the only nonzero $\tilde{X}$ to solve (2) are $\tilde{\lambda}_k = -(k\pi)^2$ and $\tilde{X}_k(x) = \cos(k\pi x)$ for $k = 0, \dots$, which concludes the proof. □

This same procedure of breaking the eigenvalue into cases $\lambda > 0$ and $\lambda \leq 0$ can also show the following

Proposition 2.2. The values $\lambda_k = -(k\pi)^2$ and $X_k(x) = \sin(k\pi x)$ for $k = 1, \dots$ are the only nonzero solutions to the Dirichlet eigenvalue problem:

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ X(0) = X(1) = 0 \end{cases} . \tag{3}$$

Proof. This will be a homework problem. □

Remark 2.1 (general procedure)**.** The general procedure for separation of variables for the following problem

$$\begin{cases} u_t(t,x) - au_{xx}(t,x) + bu_x(t,x) + cu(t,x) = 0, \quad t > 0, x \in (0,1) \\ \alpha u_x(t,0) + \beta u(t,0) = 0, \alpha u_x(t,1) + \beta u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

- Write
$$u(t,x) = \sum_{k=0}^{\infty} T_k(t) X_k(x).$$
- Solve Eigenvalue problem
$$\begin{cases} -aX_k''(x) + bX_k'(x) + cX_k(x) - \lambda_k X_k(x) = 0, \quad t > 0, x \in (0,1) \\ \alpha X_k'(0) + \beta X_k(0) = 0, \alpha X_k'(1) + \beta X_k(1) = 0 \end{cases}$$
- Compute Fourier coefficients of u_0:
$$u_0(x) = \sum_{k=0}^{\infty} c_k X_k(x), \quad c_k = \frac{\langle u_0, X_k \rangle}{\langle X_k, X_k \rangle} = \frac{\int_0^1 u_0(x) X_k(x) dx}{\int_0^1 |X_k(x)|^2 dx}$$
- Solve initial value problems for T_k:
$$T_k'(t) = \lambda_k T_k(t), \quad T_k(0) = c_k$$

Remark 2.2 (spectral methods)**.** There is a class of numerical methods, called spectral methods, that build off of separation of variables. The idea is as follows

- Write the approximate solution as
$$U^N(t,x) = \sum_{k=0}^{N} T_k(t) X_k(x)$$
where X_k is the basis from our eigenvalue problem
- Approximate the initial condition
$$u_0^N(x) = \sum_{k=0}^{N} \tilde{c}_k X_k(x)$$
- Use a time stepping method to solve the system of ODEs for T_k.

If the underlying solution is smooth, i.e. $u \in C^\infty$ has infinitely many continuous derivatives, we can expect *exponential convergence* of the method. That is, the truncation error is

$$\boldsymbol{\tau}^N = \mathcal{O}(e^{-N})$$

Additionally, the coefficients $\tilde{c}_k$ can be computed with a Fast Fourier Transform in $\mathcal{O}(N \log N)$ time (compared with solving a linear system, which is $\mathcal{O}(N^3)$ time).

Spectral methods are very efficient and accurate if the underlying solution is smooth.