COMPUTATIONAL PDE LECTURE 11

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1. Outline of today

- Explain separation of variables for different boundary conditions and right hand side

2. Separation of Variables

Last time we have considered:

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = 0, \quad t > 0, x \in (0,1) \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

and have shown that

$$u(t,x) = \sum_{k=1}^{\infty} c_k T_k(t) X_k(x)$$

solves the boundary conditions and differential equation in (1) where

$$T_k(t) = e^{-(k\pi)^2 t}, \quad X_k(t) = \sin(k\pi x).$$

and

$$c_k = \frac{\langle u_0, X_k \rangle}{\langle X_k, X_k \rangle} = \frac{\int_0^1 u_0(x)\sin(k\pi x)dx}{\int_0^1 \sin^2(k\pi x)dx} = 2\int_0^1 u_0(x)\sin(k\pi x)dx.$$

We now address what happens with more general situations.

2.1. **Right hand side f:** Suppose we need to solve.

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = f(t,x), \quad t > 0, x \in (0,1) \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{2}$$

We can reduce this into solving the 2 problems.

- w solves (2) with $f = 0$ and $u_0 \neq 0$.
- v solves (2) with $u_0 = 0$ and $f \neq 0$.

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The sum $u = v + w$ then solves (2). We have previously discussed how to solve for w. We now discuss how to solve for v.

- We write
$$v(t,x) = \sum_{k=1}^{\infty} T_k(t) X_k(x),$$
where $X_k(x) = \sin(k\pi x)$ from before.
- We also write the Fourier series for f:
$$f(t,x) \sum_{k=1}^{\infty} a_k(t) X_k(x).$$
- The heat equation now looks like
$$\sum_{k=1}^{\infty} (T_k'(t) + \pi^2 k^2 T_k(t)) X_k(x) = \sum_{k=1}^{\infty} a_k(t) X_k(x)$$
- We then solve each ODE initial value problem:
$$T_k'(t) + \pi^2 k^2 T_k(t) = a_k(t), \quad T_k(t) = 0$$
separately.

Using the example from recitation, if
$$f(t,x) = \sin(\pi x),$$
then $a_1(t) = 1$, and $a_k(t) = 0$ for all $k > 1$. Then,
$$T_1(t) = \frac{1}{\pi^2}\left(1 - e^{-\pi^2 t}\right),$$
and
$$v(t,x) = \frac{1}{\pi^2}\left(1 - e^{-\pi^2 t}\right)\sin(\pi x)$$

2.2. **Different boundary conditions.** There are two separate cases we will consider with different boundary conditions.

2.2.1. *Nonhomogenous boundary conditions.* Consider the situation where

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = f(t,x), & t > 0, x \in (0,1) \\ u(t,0) = u_\ell(t), u(t,1) = u_r(t) \\ u(0,x) = u_0(x) \end{cases} \tag{3}$$

with $u_0(0) = u_\ell(0)$ and $u_0(1) = u_r(0)$. To solve for u, we write
$$g(t,x) = u_\ell(t)(1-x) + u_r(t)x$$

and solve for $v = u - g$, where v solves

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) = f(t,x) - g_t(t,x) + \cancel{g_{xx}(t,x)}, & t > 0, x \in (0,1) \\ v(t,0) = 0, u(t,1) = 0 \\ v(0,x) = u_0(x) - g(0,x) \end{cases}$$

and apply the procedure from the previous section to solve for v. Then $u = v + g$ is the solution to (3).

2.2.2. *Neumann boundary conditions.* This section address what happens when we have homogenous Neumann boundary conditions:

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = 0, & t > 0, x \in (0,1) \\ u_x(t,0) = 0, u_x(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{4}$$

We again write the solution as

$$u(t,x) = \sum_{k=1}^{\infty} T_k(t)\tilde{X}_k(x)$$

where $\tilde{X}_k$ is a different function than $X_k(x) = \sin(k\pi x)$. This is because X_k does not satisfy the boundary conditions $X_k'(0) = X_k'(1) = 0$. Plugging u into the heat equation yields

$$\sum_{k=1}^{\infty} \left(T_k'(t)\tilde{X}_k(x) - T_k(t)\tilde{X}_k''(x) \right) = 0.$$

Solving for each individual k brings us back to

$$\frac{T_k'(t)}{T_k(t)} = \frac{X_k''(x)}{X_k(x)} = \tilde{\lambda}_k$$

where $\tilde{\lambda}_k$ is a constant. Therefore, we need to solve the eigenvalue problem

$$\begin{cases} \tilde{X}''(x) - \tilde{\lambda}\tilde{X}(x) = 0 \\ \tilde{X}'(0) = \tilde{X}'(1) = 0 \end{cases} . \tag{5}$$

We now go over how to solve these eigenvalue problems.

Proposition 2.1. The values $\tilde{\lambda}_k = -(k\pi)^2$ and $\tilde{X}_k(x) = \cos(k\pi x)$ for $k = 0, \dots$ are the only nonzero solutions to the Neumann eigenvalue problem (5).

Proof. We break the proof into 2 cases.

Case 1: Suppose $\tilde{X}, \tilde{\lambda}$ solve (5) with $\tilde{\lambda} > 0$. Then we have that $\tilde{\lambda} = a^2$ for some $a > 0$. Then the solution to $\tilde{X}'' = a^2\tilde{X}$ is

$$\tilde{X}(x) = c_1 e^{ax} + c_2 e^{-ax}.$$

Notice that $\tilde{X}'(0) = c_1 a - c_2 a$, so in order to satisfy $\tilde{X}'(0) = 0$, we require $c_1 = c_2$. If this is the case, then $\tilde{X}'(0) = c_1 a e^a - c_1 a e^{-a} = c_1 a(e^a - e^{-a})$. The only way for $c_1 a(e^a - e^{-a}) = 0$ for $a > 0$ is if $c_1 = 0$. Thus, $\tilde{X}(x) = 0$. Hence there cannot be a nonzero eigenfunction with $\tilde{\lambda} > 0$.

Case 2: Suppose $\tilde{X}, \tilde{\lambda}$ solve (5) with $\tilde{\lambda} \leq 0$. Then we have that $\tilde{\lambda} = -a^2$ for some $a \geq 0$. The solution to $\tilde{X}'' = -a^2 \tilde{X}$ is

$$\tilde{X}(x) = c_1 \cos(ax) + c_2 \sin(ax).$$

To satisfy the boundary conditions, we require

$$0 = \tilde{X}'(0) = \cancel{-ac_1 \sin(a0)} + ac_2 \cos(a0) = ac_2$$

$$0 = \tilde{X}'(0) = -ac_1 \sin(a) + ac_2 \cos(a)$$

To solve the first equation, we require $a = 0$ or $c_2 = 0$.

If $a = 0$, then

$$\tilde{X}(x) = c_1$$

is a constant. Since we do not care about the constant scaling of an eigenvector, we take $c_1 = 1$.

If $a > 0$, then $c_2 =$, and we require $-ac_1 \sin(a) = 0$. In order for $\tilde{X} \neq 0$, we need $\sin(a) = 0$, and the only solutions for $a > 0$ are $a = \pi k$ for $k = 1, \ldots$.

In conclusion, the only nonzero $\tilde{X}$ to solve (5) are $\tilde{\lambda}_k = -(k\pi)^2$ and $\tilde{X}_k(x) = \cos(k\pi x)$ for $k = 0, \ldots$, which concludes the proof. □

This same procedure of breaking the eigenvalue into cases $\lambda > 0$ and $\lambda \leq 0$ can also show the following

Proposition 2.2. The values $\lambda_k = -(k\pi)^2$ and $X_k(x) = \sin(k\pi x)$ for $k = 1, \ldots$ are the only nonzero solutions to the Dirichlet eigenvalue problem:

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ X(0) = X(1) = 0 \end{cases} . \tag{6}$$

Proof. This will be a homework problem. □