COMPUTATIONAL PDE LECTURE 10

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1. Outline of today

- Continue separation of variables

2. Separation of Variables

Last time we have considered:

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = 0, \quad t > 0, x \in (0,1) \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

and have shown that

$$u(t,x) = \sum_{k=1}^{N} c_k T_k(t) X_k(x)$$

solves the boundary conditions and differential equation in (1) where

$$T_k(t) = e^{-(k\pi)^2 t}, \quad X_k(t) = \sin(k\pi x).$$

Additionally, if

$$u_0(x) = \sum_{k=1}^{N} c_k \sin(k\pi x),$$

then u also satisfies the initial condition.

We now address what if u_0 is not the finite sum of sines.

2.1. More general initial conditions: Fourier series.

Suppose

$$u_0(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x),$$

then

$$u(t,x) = \sum_{k=1}^{N} c_k T_k(t) X_k(x)$$

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would formally solve (1). I say formal solution because we have not actually proved that the infinite series makes sense. We now show a sufficient condition for the infinite series to make sense.

Lemma 2.1 (Weierstrass M-test)**.** Let $f_k : [0,1] \to \mathbb{R}$ be a sequence of continuous functions with $M_k = \max_{x\in[0,1]} |f_k(x)|$. If

$$\sum_{k=1}^{\infty} M_k < \infty,$$

then for any x, the series

$$\sum_{k=1}^{\infty} f_k(x)$$

converges absolutely. Also, define $f(x) = \sum_{k=1}^{\infty} f_k(x)$. We also have that the series converges uniformly, i.e.

$$\lim_{N\to\infty} \max_{x\in[0,1]} \left| f(x) - \sum_{k=1}^{N} f_k(x) \right| = 0,$$

and f is continuous.

Proof. We show that the sequence $F_N = \sum_{k=1}^{N} f_k$ is uniformly Cauchy. Let $m \geq N$, and we compute

$$\big|F_N(x) - F_m(x)\big| \leq \sum_{k=N+1}^{m} |f_k(x)| \leq \sum_{k=N+1}^{m} M_k \leq \sum_{k=N+1}^{\infty} M_k$$

Taking a max over all x leads to

$$\max_{x\in[0,1]} \big|F_N(x) - F_m(x)\big| \leq \sum_{k=N+1}^{\infty} M_k$$

and taking a limit as $N \to \infty$ shows

$$\lim_{N\to\infty} \sup_{m\geq N} \max_{x\in[0,1]} \big|F_N(x) - F_m(x)\big| \leq 0$$

Hence, F_N is uniformly Cauchy, and converges uniformly to some continuous f. □

The relevant result for us is the following

Proposition 2.1. Let $\{c_k\}_{k\in\mathbb{N}}$ be a sequence and define:

$$u_0^N(x) = \sum_{k=1}^{N} c_k \sin(k\pi x)$$

If

$$\sum_{k=1}^{\infty} |c_k| < \infty,$$

then u_0^N converges uniformly on $[0,1]$ and the limit is the continuous function

$$u_0(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x).$$

2.1.1. *Computing the coefficients.* Suppose u_0 is some continuous function. How do we find c_k such that

$$u_0(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)?$$

The coefficients in this case are known as **Fourier coefficients**.

We first begin by looking at linear algebra. Given a vector $\mathbf{w} \in \mathbb{R}^n$ what is the best approximation $\mathbf{v}^* \in \mathbb{V}$, where $\mathbb{V} \subset \mathbb{R}^n$ is a subset of $\mathbb{R}^n$. We first consider the Euclidean norm

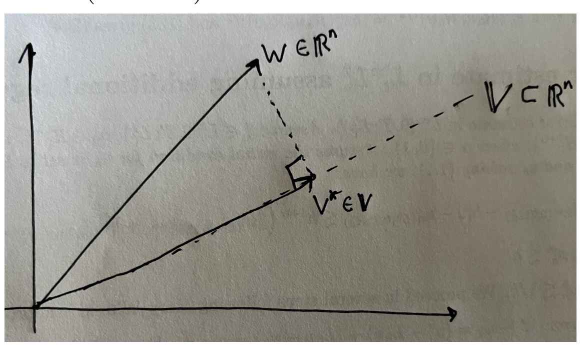
$$\|\mathbf{v}\|_2 = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

Then

$$\|\mathbf{v}^* - \mathbf{w}\|_2 = \inf_{\mathbf{u} \in \mathbb{V}} \|\mathbf{u} - \mathbf{w}\|_2$$

if and only if

$$(\mathbf{v}^* - \mathbf{w}) \cdot \mathbf{u} = 0 \text{ for all } \mathbf{u} \in \mathbb{V}.$$



To see this one direction of the if and only if, we can write

$$\begin{aligned}
\|\mathbf{v}^* - \mathbf{w}\|_2^2 &= \mathbf{v}^* \cdot (\mathbf{v}^* - \mathbf{w}) - \mathbf{w} \cdot (\mathbf{v}^* - \mathbf{w}) \\
&= -\mathbf{w} \cdot (\mathbf{v}^* - \mathbf{w}) = (\mathbf{u} - \mathbf{w}) \cdot (\mathbf{v}^* - \mathbf{w}) \\
&\leq \|(\mathbf{u} - \mathbf{w})\|_2 \|(\mathbf{v}^* - \mathbf{w})\|_2 \qquad \text{(Cauchy-Schwarz inequality)}
\end{aligned}$$

Dividing both sides by $\|(\mathbf{v}^* - \mathbf{w})\|_2$ shows that

$$\|\mathbf{v}^* - \mathbf{w}\|_2 \leq \inf_{\mathbf{u} \in \mathbb{V}} \|\mathbf{u} - \mathbf{w}\|_2.$$

The vector $\mathbf{v}^*$ is known as the **orthogonal projection** of $\mathbf{w}$ onto $\mathbb{V}$. Suppose $\{\mathbf{v}^k\}_{k=1}^m$ is an orthogonal basis for $\mathbb{V}$. That is suppose $\{\mathbf{v}^k\}_{k=1}^m$ is a basis for $\mathbb{V}$ and $\mathbf{v}^k \cdot \mathbf{v}^j = 0$ for all $j \neq k$. Then, we write

$$\mathbf{v}^* = \sum_{k=1}^m a_k \mathbf{v}^k$$

and subtract $\mathbf{w}$ and take a dot product with $\mathbf{v}^j$ to get

$$\begin{aligned} 0 = (\mathbf{v}^* - \mathbf{w}) \cdot \mathbf{v}^j &= \left(\sum_{k=1}^m a_k \mathbf{v}^k \right) \cdot \mathbf{v}^j - \mathbf{w} \cdot \mathbf{v}^j \\ &= a_k \|\mathbf{v}^j\|_2^2 - \mathbf{w} \cdot \mathbf{v}^j, \end{aligned}$$

and

$$a_k = \frac{\mathbf{w} \cdot \mathbf{v}^j}{\|\mathbf{v}^j\|_2^2}.$$

For us, we need to somehow mimic the dot product, but for functions. The relevant inner product is the L^2 inner product:

$$\langle u, v \rangle = \int_0^1 u(x)v(x)dx.$$

and L^2 norm:

$$\|u\|_{L^2[0,1]} = \sqrt{\langle u, u \rangle}.$$

Luckily, our sine basis is orthogonal in the L^2 inner product.

Lemma 2.2. Let $X_k(x) = \sin(\pi k x)$. Then

$$\langle X_k, X_j \rangle = \begin{cases} 0, & j \neq k \\ 1/2, & j = k \end{cases}.$$

We then have the following approximation result.

Proposition 2.2. Let $X_k(x) = \sin(\pi k x)$. Let u_0 be continuous. Then, we define the Fourier coefficient as

$$c_k = \frac{\langle X_k, X_j \rangle}{\langle X_k, X_k \rangle}$$

and the sum

$$u_0^N(x) = \sum_{k=1}^N c_k X_k$$

is the best L^2 approximation of u_0 in $\mathbb{V} = \text{span}\{X_k\}_{k=1}^N$. That is,

$$\int_0^1 |u_0(x) - u_0^N(x)|^2 dx = \inf_{v \in \mathbb{V}} \int_0^1 |u_0(x) - v(x)|^2 dx.$$

Moreover if the coefficients c_k satisfy

$$\sum_{k=1}^{\infty} |c_k| < \infty,$$

then $u_0^N \to u_0$ uniformly.

Remark 2.1 (estimates on Fourier coefficients)**.** The proof of this can be done using integration by parts.

If $u_0 \in C^2(0,1)$ with $u_0(0) = u_0(1) = u_0'(0) = u_0'(1) = 0$, then

$$|c_k| \leq \frac{\max_{x \in [0,1]} |u_0''(x)|}{k^2 \pi^2}$$

and

$$\sum_{k=1}^{\infty} |c_k| < \infty.$$

Ultimately, to compute a solution to the heat equation (1), we follow the following procedure.

- Compute the Fourier coefficients

$$c_k = 2 \int_0^1 u_0(x) \sin(\pi k x) dx$$

- Verify that

$$\sum_{k=1}^{\infty} |c_k| < \infty$$

- Write down

$$u(t,x) = \sum_{k=1}^{\infty} c_k e^{-(k\pi)^2 t} \sin(k \pi x)$$

and u solves (1).