COMPUTATIONAL PDE LECTURE 9

LUCAS BOUCK

1. Outline of today

- Prove Poincare inequality
- Begin separation of variables.

2. Energy estimates for the heat equation

Recall that last time we proved.

Proposition 2.1 (energy estimate)**.** Let u be a C^2 solution to the heat equation with homogenous Dirichlet boundary conditions.

$$(1) \qquad \begin{cases} u_t(t,x) - u_{xx}(t,x) = f(t,x), & t > 0 \text{ and } x \in (0,1) \\ u(0) = u(1) = 0, & \text{(boundary condition)} \\ u(0,x) = u_0(x) & \text{(initial condition)} \end{cases}$$

Then, for all $T > 0$, we have

$$(2) \quad \int_0^1 |u(T,x)|^2 dx + \int_0^T \int_0^1 |u_x(t,x)|^2 dx dt \le \int_0^1 |u_0(x)|^2 dx + \int_0^T \int_0^1 |f(t,x)|^2 dx dt$$

A corollary of the energy estimate is that solutions to the heat equation are unique.

Corollary 2.1 (uniqueness of solutions)**.** C^2 solutions to (1) are unique.

Proof. Let u_1, u_2 be C^2 solutions to (1). Then the difference $v = u_1 - u_2$ solves

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) = 0, & t > 0 \text{ and } x \in (0,1) \\ v(0) = v(1) = 0, \\ v(0,x) = 0 \end{cases}$$

Applying the energy estimates to v shows $v = 0$. □

Lemma 2.1 (Poincare's inequality)**.** Let $f \in C^1[0,1]$ satisfy $f(0) = 0$, then

$$\int_0^1 |f(x)|^2 dx \le \int_0^1 |f'(x)|^2 dx$$

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Proof. We can write

$$f(x) = \cancel{f(0)} + \int_0^x f'(y)dy$$

Taking an absolute value of both sides yields

$$|f(x)| = \left|\int_0^x f'(y)dy\right| \leq \int_0^x |f'(y)|dy \leq \int_0^1 |f'(y)|dy$$

Squaring both sides yields:

$$|f(x)|^2 \leq \left(\int_0^1 |f'(y)|dy\right)^2$$

I claim that

$$\left(\int_0^1 |f'(y)|dy\right)^2 \leq \int_0^1 |f'(y)|^2 dy.$$

Note that for two numbers a, b, we have

$$\left(\frac{1}{2}a + \frac{1}{2}b\right)^2 \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$$

because $x \mapsto x^2$ is convex. For a Riemann sum at $x_j = j/2^N$ of some function g, we have

$$\left(\sum_{j=1}^{2^N} \frac{1}{2^N} g(x_i)\right)^2 \leq \frac{1}{2}\left(\sum_{j=1}^{2^{N-1}} g(x_j)\right)^2 + \frac{1}{2}\left(\sum_{j=2^{N-1}+1}^{2^N} \frac{1}{2^{N-1}} g(x_j)\right)^2.$$

We can continue recursively to show

$$\left(\frac{1}{2^N}\sum_{j=1}^{2^N} g(x_i)\right)^2 \leq \frac{1}{2^N}\sum_{j=1}^{2N} g(x_i)^2$$

The limit of these sums as $N \to \infty$ is

$$\left(\int_0^1 g(x)dx\right)^2 \leq \int_0^1 g(x)^2 dx$$

Setting $g(x) = |f'(x)|$ proves the claim. Note that this is a special case of **Jensen's inequality**. We now insert the claim into our inequality

$$|f(x)|^2 \leq \left(\int_0^1 |f'(y)|dy\right)^2 \leq \int_0^1 |f'(y)|^2 dy$$

Taking the max over all $x \in [0,1]$ yields

$$\max_{x\in[0,1]} |f(x)|^2 \le \left(\int_0^1 |f'(y)|dy\right)^2 \le \int_0^1 |f'(y)|^2 dy$$

and

$$\int_0^1 |f(x)|^2 dx \le \max_{x\in[0,1]} |f(x)|^2 \le \int_0^1 |f'(x)|^2 dx.$$

□

Remark 2.1 (another version of Poincare)**.** Note that we technically proved a stronger version of Poincare:

$$\max_{x\in[0,1]} |f(x)|^2 \le \int_0^1 |f'(x)|^2 dx.$$

This can be used to prove an alternative stability result for Poisson's equation without using maximum principle.

3. Separation of variables

We now construct solutions to the heat equation. We start with

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = 0, \quad t > 0, x \in (0,1) \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{3}$$

The main idea of separation of variables is to look for solutions of the form

$$u(t,x) = \sum_{k=1}^N c_k u_k(t,x)$$

where we can separate the variables of each u_k:

$$u_k(t,x) = T_k(t) X_k(x).$$

Importantly, if each u_k solves the differential equation in (3), i.e.

$$\partial_t u_k(t,x) - \partial^2_{xx} u_k(t,x) = 0,$$

then u solves the differential equation because:

$$u_t(t,x) - u_{xx}(t,x) = \sum_{k=1}^N c_k \partial_t u_k(t,x) - \partial^2_{xx} u_k(t,x) = 0.$$

We have used the fact that the differential equation is linear. This is also known as the **Principle of Superposition**.

We now try to solve for each u_k. We plug in $u_k(t,x) = T_k(t)X_k(x)$ into the differential equation to get

$$\partial_t u_k(t,x) - \partial^2_{xx} u_k(t,x) = T_k'(t)X_k(x) - T_k(t)X_k''(x) = 0.$$

Rearranging the equation leads to

$$\frac{T_k'(t)}{T_k(t)} = \frac{X_k''(x)}{X_k(x)}.$$

Notice that each side of the above equation is equal for all choices of t, x. The only functions that can satisfy this are constant functions. We'll let λ_k denote the constant such that

$$\frac{T_k'(t)}{T_k(t)} = \frac{X_k''(x)}{X_k(x)} = \lambda_k,$$

and we now solve for X_k and T_k separately.

We first solve for X such that

$$X_k''(x) = \lambda_k X_k(x), \quad X_k(0) = X_k(1) = 0.$$

Note that we enforce the boundary condition on X_k in order for u_k and u to satisfy the boundary conditions. We saw in HW 1, that the possible solutions to this problem (up to a multiplying constant) are

$$X_k(x) = \sin(k\pi x), \quad \lambda_k = -k^2\pi^2.$$

We can also solve for

$$T_k'(t) = k^2\pi^2 T_k(t),$$

whose solutions are

$$T_k(t) = T_k(0)e^{-k^2\pi^2 t}.$$

Hence, our candidate solution u is

$$u(t,x) = \sum_{k=1}^{N} c_k T_k(t)X_k(x) = \sum_{k=1}^{N} c_k e^{-k^2\pi^2 t}\sin(k\pi x)$$

where c_k are constants that are to be determined, which we will do next time.