COMPUTATIONAL PDE LECTURE 8

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1. Outline of today

- Derive the heat equation
- Derive energy estimates of the heat equation

2. Derive the heat equation

We'll start with deriving the heat equation on the real line. That is the internal heat energy density $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ solves

$$u_t(t,x) - k u_{xx}(t,x) = f(t,x). \tag{1}$$

where f is a heat source or sink. We first start with a point x_0 and an interval $(x_0 - \delta, x_0 + \delta)$. We first write the **conservation of energy**

(total change in energy) = (total energy change from external sources)− (energy leaving)

which can be expressed mathematically as

$$\frac{d}{dt}\int_{x_0-\delta}^{x_0+\delta} u(t,x)dx = \int_{x_0-\delta}^{x_0+\delta} f(t,x)dx - j(t, x_0+\delta) + j(t, x_0 - \delta).$$

Here, j denotes the **heat flux**.

We do cannot determine a precise form of the heat flux a priori, but we assume the heat flux satisfies an empirical law. Specifically, we assume **Fourier's law of heat**, which states there is a constant $k > 0$ such that

$$j(t,x) = -k u_x(t,x).$$

Intuitively, this means heat flows from hot areas to cold areas. The conservation of energy now reads:

$$\frac{d}{dt}\int_{x_0-\delta}^{x_0+\delta} u(t,x)dx = \int_{x_0-\delta}^{x_0+\delta} f(t,x)dx + k u_x(t, x_0+\delta) - k u_x(t, x_0 - \delta).$$

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We now use fundamental theorem of calculus (this would be divergence theorem in higher dimensions), to write

$$ku_x(t, x_0 + \delta) - ku_x(t, x_0 - \delta) = \int_{x_0-\delta}^{x_0+\delta} ku_{xx}(t, x)dx,$$

so

$$\frac{d}{dt}\int_{x_0-\delta}^{x_0+\delta} u(t, x)dx = \int_{x_0-\delta}^{x_0+\delta} f(t, x)dx + \int_{x_0-\delta}^{x_0+\delta} ku_{xx}(t, x)dx.$$

We also apply Leibniz integral rule to the left hand side to get

$$\frac{d}{dt}\int_{x_0-\delta}^{x_0+\delta} u(t, x)dx = \int_{x_0-\delta}^{x_0+\delta} u_t(t, x)dx,$$

and

$$\int_{x_0-\delta}^{x_0+\delta} u_t(t, x)dx = \int_{x_0-\delta}^{x_0+\delta} f(t, x)dx + \int_{x_0-\delta}^{x_0+\delta} ku_{xx}(t, x)dx.$$

Rearranging and dividing both sides by 2δ leads to

$$\frac{1}{2\delta}\int_{x_0-\delta}^{x_0+\delta} u_t(t, x) - ku_{xx}(t, x)dx = \frac{1}{2\delta}\int_{x_0-\delta}^{x_0+\delta} f(t, x)dx.$$

Recall that for continuous functions:

$$\lim_{\delta\to 0}\frac{1}{2\delta}\int_{x_0-\delta}^{x_0+\delta} f(t, x)dx = f(t, x_0),$$

and we have the heat equation (1).

3. Energy estimates for the heat equation

Recall that for Poissons equation, we answered the following questions:

- Existence and construction of solutions: Green's functions
- Stability: maximum principle
- Uniqueness of solutions: maximum principle

Today, we'll address the stability as well as uniqueness of solutions to the heat equation by looking at what are called **energy estimates**.

Proposition 3.1 (energy estimate)**.** Let u be a C^2 solution to the heat equation with homogenous Dirichlet boundary conditions.

$$\begin{cases} u_t(t, x) - u_{xx}(t, x) = f(t, x), & t > 0 \text{ and } x \in (0, 1) \\ u(0) = u(1) = 0, & \text{(boundary condition)} \\ u(0, x) = u_0(x) & \text{(initial condition)} \end{cases} \tag{2}$$

Then, for all $T > 0$, we have

$$\int_0^1 |u(T,x)|^2 dx + \int_0^T \int_0^1 |u_x(t,x)|^2 dx dt \leq \int_0^1 |u_0(x)|^2 dx + \int_0^T \int_0^1 |f(t,x)|^2 dx dt \tag{3}$$

Proof. We begin by multiplying the heat equation by $u(t,x)$ and integrating from $x = 0$ to $x = 1$:

$$\int_0^1 u_t(t,x)u(t,x)dx - \int_0^1 u_{xx}(t,x)u(t,x)dx = \int_0^1 f(t,x)u(t,x)dx$$

We now make a few observations about this equation:

- The first time can be written as a pure time derivative using the chain rule and Leibniz integral rule:

$$\int_0^1 u_t(t,x)u(t,x)dx = \frac{1}{2}\int_0^1 \frac{\partial}{\partial t}|u(t,x)|^2 dx = \frac{d}{dt}\frac{1}{2}\int_0^1 |u(t,x)|^2 dx$$

- Integrating the second term by parts leads to:

$$\begin{aligned} -\int_0^1 u_{xx}(t,x)u(t,x)dx &= \cancel{-u_x(t,1)u(t,1)} + \cancel{u_x(t,0)u(t,0)} + \int_0^1 u_x(t,x)u_x(t,x)dx \\ &= \int_0^1 |u_x(t,x)|^2 dx \end{aligned}$$

  Thus, our equality is now

$$\frac{d}{dt}\frac{1}{2}\int_0^1 |u(t,x)|^2 dx + \int_0^1 |u_x(t,x)|^2 dx = \int_0^1 f(t,x)u(t,x)dx$$

- The term on the right hand side can be written in a way that can be controlled by terms on the left hand side. We first use a fact about real numbers, which is for any $a, b \in \mathbb{R}$, we have

$$(a-b)^2 \geq 0.$$

  Expanding the quadratic leads to

$$a^2 + b^2 - 2ab \geq 0.$$

  Rearranging this inequality yields what is sometimes called **Young's inequality**:

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2.$$

  Hence,

$$\int_0^1 f(t,x)u(t,x)dx \leq \frac{1}{2}\int_0^1 |f(t,x)|^2 dx + \frac{1}{2}\int_0^1 |u(t,x)|^2 dx.$$

The other useful inequality is to control the size of u with its derivative u_x, which is known as **Poincare's inequality**:

$$\int_0^1 |u(t,x)|^2 dx \leq \int_0^1 |u_x(t,x)|^2 dx,$$

which we'll prove later. Using Poincare leads us to

$$\int_0^1 f(t,x)u(t,x)dx \leq \frac{1}{2}\int_0^1 f(t,x)^2 dx + \frac{1}{2}\int_0^1 |u_x(t,x)|^2 dx.$$

Combing all the equalities and estimates yields:

$$\frac{d}{dt}\frac{1}{2}\int_0^1 |u(t,x)|^2 dx + \int_0^1 |u_x(t,x)|^2 dx \leq \frac{1}{2}\int_0^1 f(t,x)^2 dx + \frac{1}{2}\int_0^1 |u_x(t,x)|^2 dx.$$

We now absorb the the integral of $(u_x)^2$ onto the left hand side, multiply everything by 2, and integrate in time from 0 to T:

$$\int_0^T \frac{d}{dt}\int_0^1 |u(t,x)|^2 dx dt + \int_0^T \int_0^1 |u_x(t,x)|^2 dx dt \leq \int_0^T \int_0^1 f(t,x)^2 dx dt.$$

Realizing

$$\int_0^T \frac{d}{dt}\int_0^1 |u(t,x)|^2 dx dt = \int_0^1 |u(T,x)|^2 dx - \int_0^1 |u(0,x)|^2 dx$$

finishes the proof. □

A corollary of the energy estimate is that solutions to the heat equation are unique.

Corollary 3.1 (uniqueness of solutions)**.** C^2 solutions to (2) are unique.

Proof. Let u_1, u_2 be C^2 solutions to (2). Then the difference $v = u_1 - u_2$ solves

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) = 0, & t > 0 \text{ and } x \in (0,1) \\ v(0) = v(1) = 0, \\ v(0,x) = 0 \end{cases}$$

Applying the energy estimates to v shows $v = 0$. □

Lemma 3.1 (Poincare's inequality)**.** Let $f \in C^1[0,1]$ satisfy $f(0) = 0$, then

$$\int_0^1 |f(x)|^2 dx \leq \int_0^1 |f'(x)|^2 dx$$

Proof. We can write

$$f(x) = \cancel{f(0)} + \int_0^x f'(y) dy$$

Taking an absolute value of both sides yields

$$|f(x)| = \left|\int_0^x f'(y)dy\right| \leq \int_0^x |f'(y)|dy \leq \int_0^1 |f'(y)|dy$$

Squaring both sides yields:

$$|f(x)|^2 \leq \left(\int_0^1 |f'(y)|dy\right)^2$$

I claim that

$$\left(\int_0^1 |f'(y)|dy\right)^2 \leq \int_0^1 |f'(y)|^2 dy.$$

Note that for two numbers a, b, we have

$$\left(\frac{1}{2}a + \frac{1}{2}b\right)^2 \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$$

because $x \mapsto x^2$ is convex. For a Riemann sum at $x_j = j/2^N$ of some function g, we have

$$\left(\sum_{j=1}^{2^N} \frac{1}{2^N} g(x_i)\right)^2 \leq \frac{1}{2}\left(\sum_{j=1}^{2^{N-1}} g(x_j)\right)^2 + \frac{1}{2}\left(\sum_{j=2^{N-1}+1}^{2^N} \frac{1}{2^{N-1}} g(x_j)\right)^2.$$

We can continue recursively to show

$$\left(\frac{1}{2^N}\sum_{j=1}^{2^N} g(x_i)\right)^2 \leq \frac{1}{2^N}\sum_{j=1}^{2N} g(x_i)^2$$

The limit of these sums as $N \to \infty$ is

$$\left(\int_0^1 g(x)dx\right)^2 \leq \int_0^1 g(x)^2 dx$$

Setting $g(x) = |f'(x)|$ proves the claim. Note that this is a special case of **Jensen's inequality**. We now insert the claim into our inequality

$$|f(x)|^2 \leq \left(\int_0^1 |f'(y)|dy\right)^2 \leq \int_0^1 |f'(y)|^2 dy$$

Taking the max over all $x \in [0, 1]$ yields

$$\max_{x\in[0,1]} |f(x)|^2 \leq \left(\int_0^1 |f'(y)|dy\right)^2 \leq \int_0^1 |f'(y)|^2 dy$$

and

$$\int_0^1 |f(x)|^2 dx \leq \max_{x\in[0,1]} |f(x)|^2 \leq \int_0^1 |f'(x)|^2 dx.$$

□

Remark 3.1 (another version of Poincare)**.** Note that we technically proved a stronger version of Poincare:

$$\max_{x\in[0,1]} |f(x)|^2 \leq \int_0^1 |f'(x)|^2 dx.$$

This can be used to prove an alternative stability result for Poisson's equation without using maximum principle.