COMPUTATIONAL PDE LECTURE 5

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1. Outline of today

- Truncation error of our discretization of Poisson: consistency
- Convergence of the finite difference method assuming stability.

We have been studying

$$\begin{cases} -u''(x) = f(x) & \text{for } x \in (0,1) \\ u(0) = u_\ell, & \\ u(1) = u_r & \end{cases}, \tag{1}$$

for $u_\ell = u_r = 0$. The rest of the lecture will cover when u_ℓ, u_r are not necessarily 0.

2. Finite Differences Approximation of Poisson

Recall that our main tool for discretizing Poisson's equation was the finite difference approximation:

$$D_h^2 f(x) := \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}. \tag{2}$$

Using the above approximation (2), we discretized (1) with the equation at $x_j = hj$

$$-D_h^2 U^h(x_j) = -\frac{U^h(x_{j+1}) - 2U^h(x_j) + U^h(x_{j-1})}{h^2} = f(x_j),$$

and the equations

$$U^h(x_0) = u_\ell, U^h(x_N) = u_r$$

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at x_0 and x_{N+1} respectively. Combining the above two conditions leads to the following linear system of equations:

$$\underbrace{\begin{pmatrix} 1 & 0 & \dots & & \\ -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & 0 & \dots \\ & \ddots & \ddots & \ddots & \\ \dots & 0 & -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} \\ & & \dots & 0 & 1 \end{pmatrix}}_{=:\mathbf{A}^h} \underbrace{\begin{pmatrix} U^h(x_0) \\ U^h(x_1) \\ \vdots \\ U^h(x_{N-1}) \\ U^h(x_N) \end{pmatrix}}_{=:\mathbf{U}^h} = \underbrace{\begin{pmatrix} u_\ell \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \\ u_r \end{pmatrix}}_{\mathbf{f}^h}. \tag{3}$$

We now begin to analysis this method.

2.1. **Truncation error analysis: consistency.** By replacing $u''(x_j)$ with $D_h^2 U^h(x_j)$, we are making an approximation. A natural question to ask is how accurate is that approximation in terms of the number of grid points $N+1$ or the mesh size h. The first step to deriving an estimate for the error $|u(x_j) - U^h(x_j)|$ is to derive the **truncation error** $\boldsymbol{\tau}^h$, which is the error of plugging in the exact solution u into the discrete system (3):

$$\boldsymbol{\tau}^h = \mathbf{f}^h - \mathbf{A}^h \mathbf{u}$$

where $\mathbf{u}_j = u(x_j)$. Another way of writing the truncation error is $|\boldsymbol{\tau}_j^h| = |u''(x_j) - D_h^2 u(x_j)|$. We say the discrete system (3) is **consistent** with (1) if the truncation error $\boldsymbol{\tau}^h$ satisfies

$$\lim_{h\to 0} \|\boldsymbol{\tau}^h\|_\infty = 0$$

where

$$\|\boldsymbol{\tau}^h\|_\infty = \max_{0\le j\le N+1} |\boldsymbol{\tau}_j^h|$$

Proposition 2.1 (consistency and truncation error)**.** Suppose u solves is $C^4[0,1]$ and solves (1). Then for $1 \le j \le N-1$, $\boldsymbol{\tau}_j^h$ satisfies

$$|\boldsymbol{\tau}_j^h| \le \frac{h^2}{12} \max_{x\in[0,1]} |u^{(4)}(x)|$$

Proof. The proof relies heavily on Taylor expansion. For $u \in C^4[0,1]$, $x \in (0,1)$, and sufficiently small h, we have

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u^{(3)}(x) + \frac{h^4}{4!}u^{(4)}(h_1) \tag{4}$$

where $h_1 \in (x, x+h)$ and

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2!}u''(x) - \frac{h^3}{3!}u^{(3)}(x) + \frac{h^4}{4!}u^{(4)}(h_2) \tag{5}$$

$h_2 \in (x-h, x)$. Adding (4) and (5) together yields

$$u(x+h) + u(x-h) = 2u(x) + h^2 u''(x) + \frac{h^4}{4!}(u^{(4)}(h_1) + u^{(4)}(h_2)).$$

Hence,

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u''(x) + \frac{h^2}{4!}(u^{(4)}(h_1) + u^{(4)}(h_2)).$$

Subtracting $u''(x)$ from both sides and taking absolute value allows us to estimate:

$$\begin{aligned}\left|\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} - u''(x)\right| &= \left|\frac{h^2}{4!}(u^{(4)}(h_1) + u^{(4)}(h_2))\right| \\ &\le \frac{2h^2}{4!} \max_{z\in[0,1]} |u^{(4)}(z)| = \frac{h^2}{12} \max_{z\in[0,1]} |u^{(4)}(z)|\end{aligned}$$

The above estimate is true for any $x \in (0,1)$, so it is true for x_j. Hence

$$|\boldsymbol{\tau}_j^h| \le \frac{h^2}{12} \max_{z\in[0,1]} |u^{(4)}(z)|$$

for all j, which is the desired result. □

Definition 2.1 (rate of convergence)**.** A convergent sequence $z_N \to z$ as $N \to \infty$ converges with rate α if there is a $c > 0$ such that

$$|z_N - z| \le c(1/N)^\alpha$$

Indexing with h, we say a sequence $z_h \to z$ as $h \to 0$ converges with rate α if there is a $c > 0$ such that

$$|z_h - z| \le ch^\alpha.$$

Remark 2.1 (rate of convergence of truncation error)**.** We can see that $\boldsymbol{\tau}_j^h \to 0$ with rate 2.

Remark 2.2 (big O notation)**.** We say that $z_h = O(a_h)$ as $h \to 0$ if there is a $h_0, c > 0$ such that

$$z_h \le ca_h$$

for all $h < h_0$. For example, $\boldsymbol{\tau}_j^h = O(h^2)$.

2.2. **Discrete maximum principle: stability.** The rest of the discussion will assume $u_r = u_\ell = 0$ though these results can be adapted if $u_r \neq 0$ and $u_\ell \neq 0$, which will be an assignment problem for the continuous problem. The book also presents different proofs based on discrete Green's function.

So far, we have shown that the exact solution u solves the discrete equation with an additional truncation error term that is $O(h^2)$. However, we have so far not said anything about the actual error:

$$\max_{0\leq j\leq N} |u(x_j) - U^h(x_j)|.$$

What we'll need is a stability result, which we'll prove in Monday's lecture.

Proposition 2.2 (stability of the finite difference scheme)**.** For $u_\ell = u_r = 0$, the discrete scheme is stable in the sense that

$$\max_{0\leq j\leq N} |U^h(x_j)| \leq \max_{1\leq j\leq N} |f(x_j)|$$

In other words, if $\mathbf{A}^h\mathbf{U}^h = \mathbf{f}^h$, then

$$\|\mathbf{U}^h\|_\infty \leq \|\mathbf{f}^h\|_\infty$$

Additionally, the right hand side of the above inequality is independent of h.

We will prove this later. There are two important corollaries of stability. The first is that there exist unique discrete solutions U^h.

Corollary 2.1 (existence and uniqueness of discrete solutions)**.** For $u_\ell = u_r = 0$, there exists a unique $\mathbf{U}^h$ such that solves the system of linear equations (3)

Proof. Since $\mathbf{A}^h$ is a square matrix, existence of solutions to (3) is equivalent to uniqueness of solutions by the Fundamental Theorem of Linear Algebra. To show uniqueness let $\mathbf{U}^h, \mathbf{V}^h$ solve

$$\mathbf{A}^h\mathbf{U}^h = \mathbf{f}^h, \quad \mathbf{A}^h\mathbf{V}^h = \mathbf{f}^h.$$

Subtracting these two equations yields

$$\mathbf{A}^h(\mathbf{U}^h - \mathbf{V}^h) = \mathbf{0}.$$

Applying the stability result shows that

$$\|\mathbf{U}^h - \mathbf{V}^h\|_\infty \leq \|\mathbf{0}\|_\infty$$

and $\mathbf{U}^h = \mathbf{V}^h$. □

The second important corollaries is the desired error estimate:

Theorem 2.1 (convergence and error estimate)**.** Let $u_\ell = u_r = 0$. Let $u \in C^4[0,1]$ be a solution of (1) and let U^h be a solution of (3), then we have

$$\max_{0\le j\le N} |u(x_j) - U^h(x_j)| \le \frac{h^2}{12} \max_{x\in[0,1]} |u^{(4)}(x)|$$

Proof. Let $\mathbf{U}^h_j = U^h(x_j)$, which is the solution to

$$\mathbf{A}^h\mathbf{U}^h = \mathbf{f}^h.$$

Let $\mathbf{u}_j = u(x_j)$ be the vector of the exact solution u evaluated at x_j. We saw that $\mathbf{u}$ solves

$$\mathbf{A}^h\mathbf{u} = \mathbf{f}^h + \boldsymbol{\tau}^h,$$

where $\boldsymbol{\tau}^h$ is the truncation error and

$$\|\boldsymbol{\tau}^h\|_\infty \le \frac{h^2}{12} \max_{z\in[0,1]} |u^{(4)}(z)|.$$

Consider the error vector $\mathbf{e}^h = \mathbf{u} - \mathbf{U}^h$. By subtracting the equations for $\mathbf{u}$ and $\mathbf{U}^h$, we see that

$$\mathbf{A}^h\mathbf{e}^h = \boldsymbol{\tau}^h.$$

We apply the stability result to the above problem to see that

$$\max_{0\le j\le N} |u(x_j) - U^h(x_j)| = \|\mathbf{e}^h\|_\infty \le \|\boldsymbol{\tau}^h\|_\infty.$$

Recall the truncation error satisfies:

$$\|\boldsymbol{\tau}^h\|_\infty \le \frac{h^2}{12} \max_{x\in[0,1]} |u^{(4)}(x)|,$$

so

$$\max_{0\le j\le N} |u(x_j) - U^h(x_j)| \le \frac{h^2}{12} \max_{x\in[0,1]} |u^{(4)}(x)|.$$

which is the desired error estimate. □

Remark 2.3 (Lax postulate)**.** The lecture today shows an important principle of numerical methods for linear differential equations:

$$\text{Consistency} + \text{Stability} \implies \text{Convergence}$$

Another name for this is the **Lax equivalence theorem**.