COMPUTATIONAL PDE LECTURE 4

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1. Outline of today

- Finish discussion of Maximum principle
- Finite difference solution of Poisson's equation in 1D

2. Uniqueness of solution: maximum principle

This discussion is parallel to Chapter 2.1.3 in the textbook, but will provide alternative proofs. The proofs in Chapter 2.1.3 of the textbook use the Green's function.

Recall we have been studying Poisson's equation 1 dimension:

$$\begin{cases} -u''(x) = f(x) & \text{for } x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}, \tag{1}$$

and proved

Proposition 2.1 (maximum principle)**.** Let u be a twice continuously differentiable function such that

$$-u''(x) > 0 \text{ for } x \in (0,1),$$
$$u(0), u(1) \geq 0$$

Then $u(x) \geq 0$ for all $x \in [0,1]$.

A consequence of the maximum principle is that the problem is stable as in the size of the solution u can be upper bounded by the size of $|f|$.

Corollary 2.1 (stability)**.** Suppose f is continuous on $[0,1]$. Let u solve (1). Then u satisfies

$$\max_{x\in[0,1]} |u(x)| \leq \max_{x\in[0,1]} |f(x)|$$

Proof. Let $M = \max_{x\in[0,1]} |f(x)|$. We break the proof into showing an upper bound $u(x) \leq M$ and lower bound $-M \leq u(x)$.

Step 1 (upper bound on u): Let $w(x) = Mx(1-x)$. The function w satisfies

$$0 \leq w(x) \leq M \text{ and } w''(x) = -2M$$

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for all $x \in [0,1]$. We define $\tilde{u}(x) = w(x) - u(x)$. Note that $\tilde{u}(x)$ solves

$$-\tilde{u}''(x) = -w''(x) - f(x) = 2M - f(x) > 0 \text{ for } x \in (0,1)$$
$$\tilde{u}(0) = w(0) = 0 \text{ and } \tilde{u}(1) = w(1) = 0.$$

Applying maximum principle to $\tilde{u}$ means $\tilde{u}(x) \geq 0$ and

$$u(x) \leq w(x) \leq M$$

for all $x \in [0,1]$.

Step 2 (lower bound on u): To show $-M \leq u(x)$ for all $x \in [0,1]$, define

$$\hat{u}(x) = u(x) + w(x)$$

and repeat the arguments of Step 1 for $\hat{u}$. □

Corollary 2.2 (uniqueness of solutions)**.** Let u_1, u_2 be twice continuously differentiable solutions to (1). Then $u_1(x) = u_2(x)$ for all $x \in [0,1]$.

Proof. Let $v = u_1 - u_2$. Then v solves

$$-v''(x) = 0 \text{ for } x \in (0,1),$$
$$v(0) = v(1) = 0$$

Apply the stability result to v to show $v \equiv 0$. □

Remark 2.1. There are two important points the proof of stability and uniqueness.

- If a linear differential equation only has terms containing u', u'', then adding constant like C to u will not change the fact that u solves the differential equation. Though $u + C$ may not satisfy the Dirichlet or Robin BC.
- For differential equations, we have the following general pattern:

$$\text{Linear differential equation} \ + \ \text{stability} \ = \ \text{unique solutions}$$

3. Finite Differences

Time permitting, we'll will start the discussion of finite differences.

Given a function f, how do we approximate it's derivative? From calculus, we remember the limit definition of derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

which states that the tangent slope is the limit of the secant slopes.

The idea behind a finite difference is that although we may not know f', we can approximate it by fixing $h > 0$ rather than taking a limit:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

The above formula is what we call a **forward finite difference**. We could also look at

$$f'(x) \approx \frac{f(x) - f(x-h)}{h},$$

which would be a **backward finite difference**. Averaging these two approximations leads a **centered finite difference**:

$$f'(x) \approx \frac{1}{2}\left(\frac{f(x+h) - f(x)}{h} + \frac{f(x) - f(x-h)}{h}\right) = \frac{f(x+h) - f(x-h)}{2h}.$$

How might we construct a finite difference approximation for $f''(x)$? First, let's consider applying a forward finite difference to $f'(x)$:

$$f''(x) \approx \frac{f'(x+h) - f'(x)}{h}.$$

We don't know $f'(x)$ or $f'(x+h)$ but further use the approximations:

$$f'(x) \approx \frac{f(x) - f(x-h)}{h} \quad \text{(backward difference)}$$

$$f'(x+h) \approx \frac{f(x+h) - f(x)}{h} \quad \text{(backward difference)}$$

Subtracting these two approximations leads to

$$f'(x+h) - f'(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h}.$$

Further dividing by h gives us a potential approximation for the second derivative:

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} =: D_h^2 f(x). \tag{2}$$

The above approximation (2) will be our main tool for solving (1) numerically.

3.1. **Finite difference approximation of Poisson's equation.** We now apply (2) to approximate (1). We first start with a

- **mesh** or grid of $N+1$ evenly spaced points on the interval $[0,1]$ as $x_j = \frac{j}{N}$ for $i = 0, \ldots, N$.
- **mesh size** as $h = \frac{1}{N}$.
- $U : \{x_j\}_{j=0}^{N+1} \to \mathbb{R}$ is an unknown **grid function** that will approximate u.

At a point x_j for $0 < j < N+1$, we replace $u''(x_j)$ with $D_h^2 U(x_j)$. As a result, we replace

$$-u''(x_j) = f(x_j)$$

with

$$-D_h^2 U(x_j) = -\frac{U(x_{j+1}) - 2U(x_j) + U(x_{j-1})}{h^2} = f(x_j)$$

At x_0, x_{N+1}, we replace the boundary condition

$$u(0), u(1) = 0$$

with

$$U(x_0), U(x_N) = 0.$$

Combining the above two conditions leads to the following linear system of equations:

$$\underbrace{\begin{pmatrix} 1 & 0 & \dots & & \\ -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & 0 & \dots \\ & \ddots & \ddots & \ddots & \\ \dots & 0 & -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} \\ & & \dots & 0 & 1 \end{pmatrix}}_{=:\mathbf{A}} \underbrace{\begin{pmatrix} U(x_0) \\ U(x_1) \\ \vdots \\ U(x_{N-1}) \\ U(x_N) \end{pmatrix}}_{=:\mathbf{U}} = \underbrace{\begin{pmatrix} 0 \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \\ 0 \end{pmatrix}}_{\mathbf{f}}. \tag{3}$$

We can then solve $\mathbf{AU} = \mathbf{f}$ using a linear algebra solver like Gaussian elimination.

3.2. **Truncation error analysis: consistency.** By replacing $u''(x_j)$ with $D_h^2 U(x_j)$, we are making an approximation. A natural question to ask is how accurate is that approximation in terms of the number of grid points $N + 1$ or the mesh size h. The first step to deriving an estimate for the error $|u(x_j) - U(x_j)|$ is to derive the **truncation error** τ^h, which is the error of plugging in the exact solution u into the discrete system (3):

$$\tau^h = \mathbf{f} - \mathbf{Au}$$

where $\mathbf{u}_j = u(x_j)$. Another way of writing the truncation error is $|\tau_j^h| = |u''(x_j) - D_h^2 u(x_j)|$. We say the discrete system (3) is **consistent** with (1) if the truncation error τ^h satisfies

$$\lim_{h \to 0} \|\tau^h\|_\infty = 0$$

where

$$\|\tau^h\|_\infty = \max_{0 \le j \le N+1} |\tau_j^h|$$

Proposition 3.1 (consistency and truncation error)**.** Suppose u solves is $C^4[0,1]$ and solves (1). Then for $1 \le j \le N$, τ_j^h satisfies

$$|\tau_j^h| \le \frac{h^2}{12} \max_{x \in [0,1]} |u^{(4)}(x)|$$

Proof. The proof relies heavily on Taylor expansion. For $u \in C^4[0,1]$, $x \in (0,1)$, and sufficiently small h, we have

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u^{(3)}(x) + \frac{h^4}{4!}u^{(4)}(h_1) \tag{4}$$

where $h_1 \in (x, x+h)$ and

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2!}u''(x) - \frac{h^3}{3!}u^{(3)}(x) + \frac{h^4}{4!}u^{(4)}(h_2) \tag{5}$$

$h_2 \in (x-h, x)$. Adding (4) and (5) together yields

$$u(x+h) + u(x-h) = 2u(x) + h^2u''(x) + \frac{h^4}{4!}(u^{(4)}(h_1) + u^{(4)}(h_2)).$$

Hence,

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u''(x) + \frac{h^2}{4!}(u^{(4)}(h_1) + u^{(4)}(h_2)).$$

Subtracting $u''(x)$ from both sides and taking absolute value allows us to estimate:

$$\begin{aligned}\left|\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} - u''(x)\right| &= \left|\frac{h^2}{4!}(u^{(4)}(h_1) + u^{(4)}(h_2))\right| \\ &\leq \frac{2h^2}{4!}\max_{z\in[0,1]}|u^{(4)}(z)| \\ &= \frac{h^2}{12}\max_{z\in[0,1]}|u^{(4)}(z)|\end{aligned}$$

The above estimate is true for any $x \in (0,1)$, so it is true for x_j. Hence

$$|\tau_j^h| \leq \frac{h^2}{12}\max_{z\in[0,1]}|u^{(4)}(z)|$$

for all j, which is the desired result. □

Definition 3.1 (rate of convergence)**.** A convergent sequence $z_N \to z$ as $N \to \infty$ converges with rate α if there is a $c > 0$ such that

$$|z_N - z| \leq c(1/N)^\alpha$$

Indexing with h, we say a sequence $z_h \to z$ as $h \to 0$ converges with rate α if there is a $c > 0$ such that

$$|z_h - z| \leq ch^\alpha.$$

Remark 3.1 (rate of convergence of truncation error)**.** We can see that $\tau_j^h \to 0$ with rate 2.