COMPUTATIONAL PDE LECTURE 3

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1. Outline of today

- Finish discussion of Green's functions
- Maximum principle

2. Green's Functions

Recall we have been studying Poisson's equation 1 dimension:

$$\begin{cases} -u''(x) = f(x) & \text{for } x \in (0,1), \\ u(0) = u(1) = 0 \end{cases}. \tag{1}$$

We also showed that if u solves (1), then u takes the form

$$u(x) = \int_0^1 G(x,y) f(y) dy \tag{2}$$

where

$$G(x,y) = \big[x(1-y) - \max\{(x-y), 0\}\big] = \begin{cases} x(1-y), & x < y \\ y(1-x) & y \le x \end{cases} \tag{3}$$

is known as a Green's function.

Note that our derivation assumed we had a solution of (1) to start. We now will double check that (2) is indeed a solution of (1).

A useful tool for showing this is **Leibniz integral rule**.

Lemma 2.1 (Leibniz integral rule)**.** Let $a, b : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Then

$$\frac{d}{dx} \int_{a(x)}^{b(x)} g(x,y) dy = g(x, b(x)) b'(x) - g(x, a(x)) a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} g(x,y) dy. \tag{4}$$

Date: September 1, 2023.

Remark 2.1 (special cases of Leibniz integral rule)**.** One special case of this rule is a version of fundamental theorem of calculus. If a is constant $b(x) = x$ and g does not depend on x, then

$$\frac{d}{dx}\int_a^x g(y)dy = g(x),$$

Another special case of Leibniz integral rule is if a, b do not depend on x, then,

$$\frac{d}{dx}\int_a^b g(x,y)dy = \int_a^b \frac{\partial}{\partial x} g(x,y)dy, \tag{5}$$

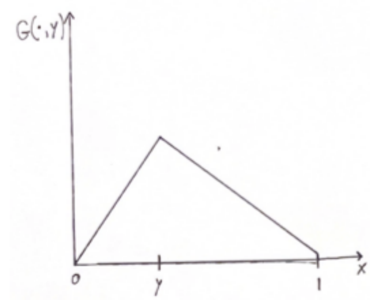
which we'll use when we study the heat equation.

We'll be unrigorous and apply (5) twice to u in (2) and compute:

$$u'(x) = \int_0^1 \frac{\partial}{\partial x}G(x,y)f(y)dy \quad \text{and} \quad u''(x) = \int_0^1 \frac{\partial^2}{\partial x^2}G(x,y)f(y)dy$$

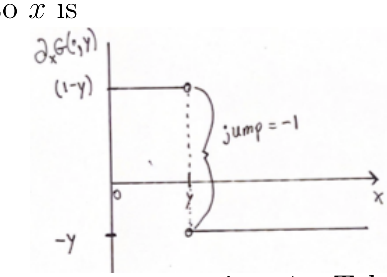
Recall that G is

$$G(x,y) = \begin{cases} x(1-y), & x < y \\ y(1-x) & y \le x \end{cases}.$$



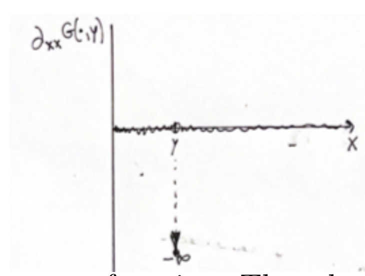
The partial derivative of G with respect to x is

$$\frac{\partial}{\partial x}G(x,y) = \begin{cases} (1-y) & x < y \\ \text{undefined} & y = x \\ y & y < x \end{cases}.$$



Notice that the jump from in $\partial_x G$ from $x < y$ to $y < x$ is -1. Taking another derivative G with respect to x is

$$\frac{\partial^2}{\partial x^2}G(x,y) = -\delta(x-y) = \begin{cases} 0 & x < y \\ -\infty & y = x \\ 0 & y < x \end{cases}.$$



Here, δ is a **Dirac delta function** and is not a true function. Though we define its integral with a continuous function f as

$$\int_0^1 \delta(x-y)f(y)dy = f(x).$$

We then have

$$-u''(x) = -\int_0^1 \underbrace{\frac{\partial^2}{\partial x^2}G(x,y)}_{=-\delta(x-y)} f(y)dy = f(x).$$

This can all be made more rigorous by splitting the integral of

$$u(x) = \int_0^1 G(x,y)f(y)dy$$

into

$$u(x) = \int_0^1 G(x,y)f(y)dy = \int_0^x y(1-x)f(y)dy + \int_x^1 x(1-y)f(y)dy$$

and apply Leibniz integral rule to each piece.

Proposition 2.1. Let $f : [0,1] \to \mathbb{R}$ be continuous on $[0,1]$, then u as defined in (2) is twice continuously differentiable and solves (1)

Proof. We split the proof into two steps.
Step 1: boundary condition: Check that $u(0) = u(1) = 0$ in the next homework.
Step 2: differential equation:
To verify $-u''(x) = f(x)$ for $0 < x < 1$, write

$$u(x) = \int_0^1 G(x,y)f(y)dy = \int_0^x y(1-x)f(y)dy + \int_x^1 x(1-y)f(y)dy$$

and apply Leibniz integral rule. We then compute

$$\frac{d}{dx}\int_0^x y(1-x)f(y)dy = x(1-x)f(x) + \int_0^x -yf(y)dy,$$
$$\frac{d}{dx}\int_x^1 x(1-y)f(y)dy = -x(1-x)f(x) + \int_x^1 (1-y)f(y)dy.$$

Adding the above terms together shows

$$u'(x) = \int_0^x -yf(y)dy + \int_x^1 (1-y)f(y)dy$$

The rest of the computation will be part of the homework. $\square$

3. Uniqueness of solution: maximum principle

This discussion is parallel to Chapter 2.1.3 in the textbook, but will provide alternative proofs. The proofs in Chapter 2.1.3 of the textbook use the Green's function.

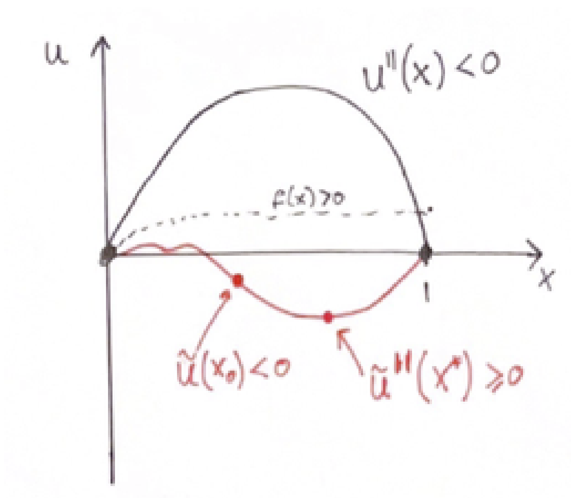
We'll now prove what is known as the weak maximum principle, which is the following statement.

Proposition 3.1 (maximum principle)**.** Let u be a twice continuously differentiable function such that

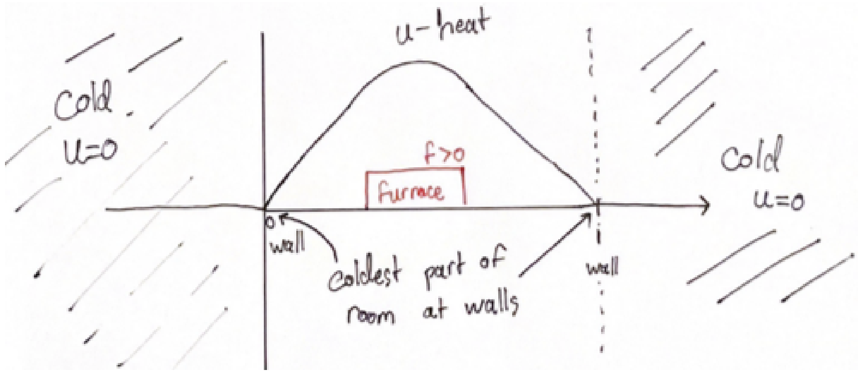
$$\begin{aligned} -u''(x) > 0 \text{ for } x \in (0,1), \\ u(0), u(1) \geq 0 \end{aligned}$$

Then $u(x) \geq 0$ for all $x \in [0,1]$.

Picture of maximum principle: Since $-u''(x) > 0$, we have that $u''(x) < 0$ for all x, and u must be concave down on $[0,1]$. The picture of the function is below in black. We now suppose that a red function $\tilde{u}$ solves the above problem and has $\tilde{u}(x_0) < 0$. Since $\tilde{u}(0), \tilde{u}(1) \geq 0$, then $\tilde{u}$ must turn upwards at some point and thus be concave up, which would contradict $\tilde{u}''(x) < 0$.



Physical intuition of maximum principle: Poisson's equation can also describe the distribution of heat in a room. If u stands for heat, then $u(0), u(1)$ are the outside temperature. If $-u''(x) = f(x)$, the f stands for the heat source, like a furnace. If $f(x) > 0$ everywhere, then we are heating the room from the inside. Maximum principle states that the coldest part of the room must be the walls.



Proof. (By contradiction). The proof follows the picture. Suppose there is an x_0 such that $u(x_0) < 0$. Then, there must be an interior minimum. That is there must be an x^* such that $u(x^*) \leq u(y)$ for all y. Consequently, $u''(x^*) \geq 0$ and $-u''(x^*) \leq 0$. However, $-u''(x^*) > 0$, hence we have a contradiction and there cannot be an x_0 such that $u(x_0) < 0$. □

A consequence of the maximum principle is that the problem is stable as in the size of the solution u can be upper bounded by the size of $|f|$.

Corollary 3.1 (stability)**.** Suppose f is continuous on $[0, 1]$. Let u solve (1). Then u satisfies

$$\max_{x\in[0,1]} |u(x)| \leq \max_{x\in[0,1]} |f(x)|$$

Proof. Let $M = \max_{x\in[0,1]} |f(x)|$. We break the proof into showing an upper bound $u(x) \leq M$ and lower bound $-M \leq u(x)$.
Step 1 (upper bound on u): Let $w(x) = Mx(1 - x)$. The function w satisfies

$$0 \leq w(x) \leq M \text{ and } w''(x) = -2M$$

We define $\tilde{u}(x) = w(x) - u(x)$. Note that $\tilde{u}(x)$ solves

$$-\tilde{u}''(x) = -w''(x) - f(x) = 2M - f(x) > 0 \text{ for } x \in (0, 1)$$
$$\tilde{u}(0) = w(0) = 0 \text{ and } \tilde{u}(1) = w(1) = 0.$$

Applying maximum principle to $\tilde{u}$ means $\tilde{u}(x) \geq 0$ and

$$u(x) \leq w(x) \leq M$$

for all $x \in [0, 1]$.
Step 2 (lower bound on u): To show $-M \leq u(x)$ for all $x \in [0, 1]$, define

$$\hat{u}(x) = u(x) + w(x)$$

and repeat the arguments of Step 1 for $\hat{u}$. Note that $\hat{u}(x)$ solves

$$-\hat{u}''(x) = -u''(x) - w''(x) = f(x) + 2M > 0 \text{ for } x \in (0,1)$$
$$\hat{u}(0) = w(0) = 0 \text{ and } \hat{u}(1) = w(1) = 0.$$

We can then apply maximum principle to $\hat{u}$ to get for all $x \in [0,1]$:

$$\hat{u}(x) \geq 0$$

and

$$u(x) \geq -w(x) \geq -M,$$

which is the desired lower bound. □

Corollary 3.2 (uniqueness of solutions)**.** Let u_1, u_2 be twice continuously differentiable solutions to (1). Then $u_1(x) = u_2(x)$ for all $x \in [0,1]$.

Proof. Let $v = u_1 - u_2$. Then v solves

$$-v''(x) = 0 \text{ for } x \in (0,1),$$
$$v(0) = v(1) = 0$$

Applying the stability result to v shows

$$\max_{x\in[0,1]} |v(x)| \leq 0$$

Hence, $v(x) = 0$ for all $x \in [0,1]$ and $u_1 = u_2$. □

Remark 3.1. There are two important points the proof of stability and uniqueness.

- If a linear differential equation only has terms containing u', u'', then adding constant like C to u will not change the fact that u solves the differential equation. Though $u + C$ may not satisfy the Dirichlet or Robin BC.
- For differential equations, we have the following general pattern:

$$\text{Linear differential equation} \ + \ \text{stability} \ = \ \text{unique solutions}$$