COMPUTATIONAL PDE LECTURE 6

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1. Outline of today

- Discrete maximum principle and stability
- Handling Neumann boundary conditions

We have been studying

$$\begin{cases} -u''(x) = f(x) & \text{for } x \in (0,1) \\ u(0) = u_\ell, \\ u(1) = u_r \end{cases}, \tag{1}$$

for $u_\ell = u_r = 0$.

2. Finite Differences Approximation of Poisson

Recall that our main tool for discretizing Poisson's equation was the finite difference approximation:

$$D_h^2 f(x) := \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}. \tag{2}$$

Using the above approximation (2), we discretized (1) with the equation at $x_j = hj$

$$-D_h^2 U^h(x_j) = -\frac{U^h(x_{j+1}) - 2U^h(x_j) + U^h(x_{j-1})}{h^2} = f(x_j),$$

and the equations

$$U^h(x_0) = u_\ell, U^h(x_N) = u_r$$

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at x_0 and x_{N+1} respectively. Combining the above two conditions leads to the following linear system of equations:

$$\underbrace{\begin{pmatrix} 1 & 0 & \dots & & \\ -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & 0 & \dots \\ & \ddots & \ddots & \ddots & \\ \dots & 0 & -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} \\ & & \dots & 0 & 1 \end{pmatrix}}_{=:\mathbf{A}^h} \underbrace{\begin{pmatrix} U^h(x_0) \\ U^h(x_1) \\ \vdots \\ U^h(x_{N-1}) \\ U^h(x_N) \end{pmatrix}}_{=:\mathbf{U}^h} = \underbrace{\begin{pmatrix} u_\ell \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \\ u_r \end{pmatrix}}_{\mathbf{f}^h}. \tag{3}$$

What we'll need is a stability result, which we'll prove in Monday's lecture.

Proposition 2.1 (stability of the finite difference scheme)**.** For $u_\ell = u_r = 0$, the discrete scheme is stable in the sense that

$$\max_{0\le j\le N} |U^h(x_j)| \le \max_{1\le j\le N} |f(x_j)|$$

In other words, if $\mathbf{A}^h\mathbf{U}^h = \mathbf{f}^h$, then

$$\|\mathbf{U}^h\|_\infty \le \|\mathbf{f}^h\|_\infty$$

Additionally, the right hand side of the above inequality is independent of h.

The important consequence of stability was the error estimate:

Theorem 2.1 (convergence and error estimate)**.** Let $u_\ell = u_r = 0$. Let $u \in C^4[0,1]$ be a solution of (1) and let U^h be a solution of (3), then we have

$$\max_{0\le j\le N} |u(x_j) - U^h(x_j)| \le \frac{h^2}{12} \max_{x\in[0,1]} |u^{(4)}(x)|$$

2.0.1. *Discrete maximum principle and proof of stability.*

Lemma 2.1 (discrete maximum principle)**.** Let U^h be the discrete solution to (3), and assume $u_r, u_\ell \ge 0$ and $f(x) > 0$ for all $x \in [0,1]$. Then, $U^h(x_j) \ge 0$ for all $j = 0, \dots, N$.

Proof. The proof follows similarly to that of the continuous problem. We proceed by contradiction. Suppose there is an $j_0 \in \{1, \dots, N-1\}$ such that $U^h(x_{j_0}) < 0$. There then exists a $J \in \{1, \dots, N-1\}$ such that $U^h(x_J) < 0$ and $U^h(x_J) \le U^h(x_j)$ for all $j \in \{0, \dots, N\}$. We now look compute $-D_h^2 U^h(x_J)$:

$$-D_h^2 U^h(x_J) = \frac{-U^h(x_{J+1}) + 2U^h(x_J) - U^h(x_{J-1})}{h^2}.$$

Since $U^h(x_J) \le U^h(x_{J\pm1})$, we have

$$\frac{-U^h(x_{J+1}) + 2U^h(x_J) - U^h(x_{J-1})}{h^2} \le \frac{-U^h(x_{J+1}) + U^h(x_{J+1}) + U^h(x_{J-1}) - U^h(x_{J-1})}{h^2} = 0.$$

Hence, $-D_h^2 U^h(x_J) \leq 0$. By assumption, $-D_h^2 U^h(x_J) = f(x_J) > 0$. Thus, $0 < -D_h^2 U^h(x_J) \leq 0$, which is a contradiction. □

In fact, the discrete maximum principle is much more general than just Poisson's equation which is part of the homework.

We can prove the desired stability result.

Proof of discrete stability. The proof also follows similarly to the proof of stability of the continuous problem. We first define $M = \max_{x\in[0,1]} |f(x)|$.

Let $\tilde{U}^h : \{x_j\}_{j=0}^N$ be defined by

$$W^h(x_j) = (M + \varepsilon)x_j(1 - x_j)$$

for $\varepsilon > 0$. One can check that for $0 \leq x_j \leq 1$:

$$D_h^2 W^h(x_j) = -(2M + 2\varepsilon) \text{ and } 0 \leq W^h(x_j) \leq M + \varepsilon.$$

We repeat the arguments of the stability result for the continuous Poisson equation.

Step 1. (upper bound) We define $\tilde{U}^h = W^h - U^h$. Then,

$$-D_h^2 \tilde{U}^h(x_j) = -D_h^2 W^h(x_j) + D_h^2 U^h(x_j) = (2M + 2\varepsilon) - f(x_j) \geq M + \varepsilon > 0,$$

and

$$\tilde{U}^h(0) = W^h(0) = 0.$$

Applying the discrete maximum principle yields

$$0 \leq \tilde{U}^h(x_j) \implies U^h(x_j) \leq W^h(x_j) \leq M + \varepsilon.$$

Step 2. (lower bound) We define $\hat{U}^h = W^h + U^h$. Then, apply the arguments of Step 1 to get that $U^h(x_j) \geq -M - \varepsilon$.

Note that we have shown $|U^h(x_j)| \leq M + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we can take a limit $\varepsilon \to 0$ to get $|U^h(x_j)| \leq M$, which is the desired stability result. □