COMPUTATIONAL PDE LECTURE 23

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1. Outline of this lecture

- Continue with discussion of upwinding

2. Setup

We are trying to solve the transport equation on the whole real line:

$$\begin{cases} u_t(t,x) + cu_x(t,x) = f(t,x), \quad t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

We will assume in this section that $c > 0$, meaning the wind is going left to right. The upwind scheme is a Forward Euler method with a backward finite difference in space:

$$D_\tau \mathbf{U}_j^n + c\left(\frac{\mathbf{U}_j^{n-1} - \mathbf{U}_{j-1}^{n-1}}{h}\right) = \mathbf{f}_j^{n-1}.$$

Essentially, this scheme "looks upwind" to compute the solution to the transport equation.

3. Why upwinding "works"

There are two reasons why the above technique works. One is that upwinding introduces some numerical diffusion, and the second is that upwinding respects conservation of mass.

3.1. Numerical Diffusion.

We again look at the upwinding scheme:

$$D_\tau \mathbf{U}_j^n + c\left(\frac{\mathbf{U}_j^{n-1} - \mathbf{U}_{j-1}^{n-1}}{h}\right) = \mathbf{f}^{n-1},$$

and observe that

$$\mathbf{U}_j^{n-1} - \mathbf{U}_{j-1}^{n-1} = \frac{1}{2}(\mathbf{U}_{j+1}^{n-1} - \mathbf{U}_{j-1}^{n-1}) + \frac{1}{2}\left(-\mathbf{U}_{j+1}^{n-1} + 2\mathbf{U}_j^{n-1} - \mathbf{U}_{j-1}^{n-1}\right).$$

Date: October 30, 2023.

Rearranging the upwind scheme leads to

$$D_\tau \mathbf{U}_j^n + \frac{c}{2h}(\mathbf{U}_{j+1}^{n-1} - \mathbf{U}_{j-1}^{n-1}) - \frac{h}{2h^2}\left(-\mathbf{U}_{j+1}^{n-1} + 2\mathbf{U}_j^{n-1} - \mathbf{U}_{j-1}^{n-1}\right) = \mathbf{f}^{n-1}.$$

Note that

$$\frac{1}{2h}(\mathbf{U}_{j+1}^{n-1} - \mathbf{U}_{j-1}^{n-1})$$

is a centered finite difference approximation of $u_x(t_{n-1}, x_j)$, and

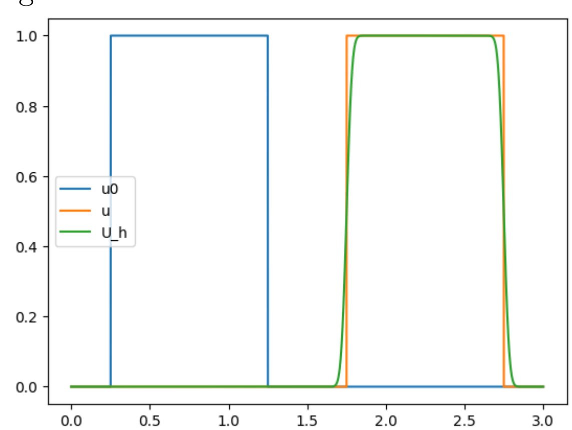
$$\frac{-\mathbf{U}_{j+1}^{n-1} + 2\mathbf{U}_j^{n-1} - \mathbf{U}_{j-1}^{n-1}}{h^2}$$

is our typical finite difference approximation for $-u_{xx}(t_n, x_j)$. Hence, the upwind scheme is numerically solving a modified heat equation:

$$u_t + cu_x - \frac{ch}{2}u_{xx} = f.$$

where the coefficient of diffusion goes to zero with $h \to 0$. From recitation, you can see that with a rough initial condition, the upwinding method adds some smoothing, the discrete solution looks similar to the true solution, but is smoother. All the methods we study will introduce some numerical diffusion.

FIGURE 1. Initial condition in blue. Exact solution in orange and numerical solution in green. Notice that the numerical method introduces some smoothing of the solution.



3.2. **Conservation of mass at numerical level.** Recall that the transport equation with $f = 0$ has a conservation of mass property:

$$\frac{d}{dt}\int_a^b u(t,x)dx = -cu(t,b) + cu(t,a).$$

where cu was a mass flux. The upwind scheme also enjoys a similar property at the discrete level if $f = 0$.

Let

$$M_h(t_n) = h\sum_{j=a}^{b} \mathbf{U}_j^n.$$

Then,

$$D_\tau M_h(t_n) = h\left(\sum_{j=a}^{b} D_\tau \mathbf{U}_j^n\right)$$

We then use the upwind scheme iteration $D_\tau \mathbf{U}_j^n = -c\frac{\mathbf{U}_j^{n-1} - \mathbf{U}_{j-1}^{n-1}}{h}$ to further expand

$$D_\tau M_h(t_n) = -c\left(\sum_{j=a}^{b} \mathbf{U}_j^{n-1} - \mathbf{U}_{j-1}^{n-1}\right) = -c\mathbf{U}_b^{n-1} + c\mathbf{U}_{a-1}^{n-1}$$

Thus, the upwind scheme mimics the conservation of mass property at the discrete level. Schemes that do this are known as **conservative**.

Remark 3.1 (upwind for nonlinear conservation laws)**.** In fact, upwinding can be a good scheme for nonlinear conservation laws of the form.

$$u_t + \partial_x(f(u)) = 0.$$

Recall the above equation has the following conservation property:

$$\frac{d}{dt}\int_a^b u(t,x)dx = -f(u(t,b)) + f(u(t,a)).$$

Assuming $f'(u) \geq 0$ for all u, the following upwind scheme is a reasonable choice for a nonlinear conservation law:

$$D_\tau \mathbf{U}_j^n + \left(\frac{f(\mathbf{U}_j^{n-1}) - f(\mathbf{U}_{j-1}^{n-1})}{h}\right) = 0$$

because it is also conservative in the sense that

$$D_\tau\left(h\sum_{j=a}^{b} D_\tau \mathbf{U}_j^n\right) = -f(\mathbf{U}_b^{n-1}) + f(\mathbf{U}_a^{n-1}),$$

so the upwind scheme outlined here is a reasonable choice for Burger's equation $f(u) = \frac{u^2}{2}$ as long as $u_0 \geq 0$.

4. Other methods for transport

Upwinding is fine, but lower order. We now discuss some strategies to get higher order methods.

4.1. **Centered Difference is unstable.** A natural next scheme to achieve higher accuracy in space would be to replace the upwind term with a centered finite difference. The resulting iteration is

$$D_\tau \mathbf{U}_j^n + c\left(\frac{\mathbf{U}_{j+1}^{n-1} - \mathbf{U}_{j-1}^{n-1}}{2h}\right) = \mathbf{f}_j^{n-1}. \tag{2}$$

However, this scheme is always unstable! The reason is that a backward or forward difference introduces numerical diffusion (think of backward Euler for the heat equation), while a centered difference does not introduce extra dissipation (think of Crank Nicholson for the heat equation). We can prove this statement more directly with von Neumann analysis.

Proposition 4.1 (instability of centered difference scheme)**.** Suppose $\mathbf{f}_j^{n-1} = 0$. The symbol of the centered difference scheme in (2) is

$$S(\xi h) = 1 - \frac{ic\tau}{h}\sin(\xi h).$$

In particular, we have

$$|S(\pi/2)| > 1,$$

so the centered difference scheme in (2) is always unstable.

Proof. The computation is an exercise. The procedure is We compute the symbol by replacing $\mathbf{U}_j^{n-1} = \mathbf{v}_j = e^{i\xi hj}$ and $\mathbf{U}_j^n = S(\xi h)\mathbf{v}_j$, and then we isolate the symbol. The key step you need is the space derivative term

$$\begin{aligned}\mathbf{v}_{j+1} - \mathbf{v}_{j-1} &= e^{i\xi h(j+1)} - e^{i\xi h(j-1)} = e^{i\xi hj}\left(e^{i\xi h} - e^{-i\xi h}\right) \\ &= \mathbf{v}_j\left(\cancel{\cos(\xi h)} + i\sin(\xi h) \cancel{- \cos(\xi h)} + i\sin(\xi h)\right) \\ &= 2i\sin(\xi h)\mathbf{v}_j.\end{aligned}$$

□

4.2. **Another scheme: Fredrich's Scheme.** In order to save the centered difference, we now look at a way to "add diffusion" or "smooth out" the numerical solution. One way to smooth out is to either add a diffusion term directly or by taking an average. The Frederich's scheme works by taking an average :

$$\frac{1}{\tau}\left[\mathbf{U}_j^n - \left(\frac{\mathbf{U}_{j-1}^{n-1} + \mathbf{U}_{j+1}^{n-1}}{2}\right)\right] + c\left(\frac{\mathbf{U}_{j+1}^{n-1} - \mathbf{U}_{j-1}^{n-1}}{2h}\right) = \mathbf{f}_j^{n-1}. \tag{3}$$

One can show that this scheme is stable and is a second order scheme (in h for fixed τ) for the following modified PDE:

$$u_t + cu_x - \frac{h^2}{2\tau}u_{xx} = 0.$$

However, note that we want to take $\tau \leq \frac{h}{c}$, so if $\tau = \frac{h}{c}$ and $h \to 0$, we no longer have a second order scheme due to the $\frac{h^2}{2\tau} \approx h$ numerical diffusion term.

We now state the specific results for the Frederich's scheme in (3) but leave the proofs as exercises.

Proposition 4.2 (stability of Frederich's)**.** Let $\mathbf{U}^n$ solve the Frederich's iteration in (3). Assume the CFL condition $\tau \leq \frac{h}{c}$. Then,

$$\|\mathbf{U}^n\|_\infty \leq \|\mathbf{U}^{n-1}\|_\infty + \tau\|\mathbf{f}^{n-1}\|_\infty.$$

Proposition 4.3 (consistency of Frederich's)**.** Let $\mathbf{u}_j^n = u(t_n, x_j)$ be the exact solution of the transport equation. Then,

$$\frac{1}{\tau}\left[\mathbf{u}_j^n - \left(\frac{\mathbf{u}_{j-1}^{n-1} + \mathbf{u}_{j+1}^{n-1}}{2}\right)\right] + c\left(\frac{\mathbf{u}_{j+1}^{n} - \mathbf{u}_{j-1}^{n}}{2h}\right) = \mathbf{f}_j^{n-1} + \boldsymbol{\tau}_j^{n-1}.$$

where there is a $C > 0$ independent of h, τ such that

$$|\boldsymbol{\tau}_j^{n-1}| \leq C\left(h^2|u_{xxx}|_{max} + \tau|u_{tt}|_{max} + \frac{h^2}{\tau}|u_{tt}|_{max}\right)$$

Proposition 4.4 (error estimate of Frederich's)**.** Let $\mathbf{U}^n$ solve the Frederich's iteration in (3). Assume the CFL condition $\tau \leq \frac{h}{c}$. Let $\mathbf{u}_j^n = u(t_n, x_j)$ be the exact solution of the transport equation. Then, the error satisfies

$$\|\mathbf{u}^n - \mathbf{U}^n\|_\infty \leq Ct_n\left(h^2|u_{xxx}|_{max} + \tau|u_{tt}|_{max} + \frac{h^2}{\tau}|u_{xx}|_{max}\right).$$