COMPUTATIONAL PDE LECTURE 22

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1. Outline of this lecture

- Start numerics for transport equation

2. Setup

We are trying to solve the transport equation on the whole real line:

$$\begin{cases} u_t(t,x) + cu_x(t,x) = f(t,x), & t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

where $c \neq 0$ is some constant speed. We consider the following numerical setup similar to our study of von Neumann stability analysis.

- meshsize $h > 0$
- time stepsize $\tau > 0$
- Uniformly spaced but infinite grid: $\{x_j\}_{j\in\mathbb{Z}}$ where $x_j = jh$
- Grid function $U^{h,\tau} : \{x_j\}_{j\in\mathbb{Z}} \to \mathbb{R}$. For simplicity, we denote

$$\mathbf{U}_j^n = U^{h,\tau}(t_n, x_j)$$

Also, recall the discrete time derivative

$$D_\tau \mathbf{U}_j^n = \frac{\mathbf{U}_j^n - \mathbf{U}_j^{n-1}}{\tau}.$$

Our goals for designing a numerical method again follow the Lax postulate, which states:

$$\text{stability} + \text{consistency} \implies \text{convergence}$$

We will easily have consistency due to Taylor expansions. Our main focus will be to design a stable method. In particular, we want the method to satisfy

$$\|\mathbf{U}^n\|_\infty \leq \|\mathbf{U}^{n-1}\|_\infty + \tau\|\mathbf{f}^{n-1}\|_\infty,$$

which mimics the stability we saw for the transport equation at the continuous level.

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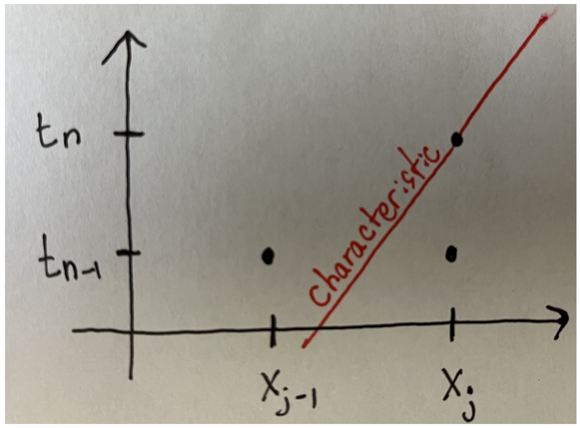

FIGURE 1. Red is the true characteristic and the black dots are the numerical stencil. We can see that the stencil "looks upwind".

3. First method: upwind scheme

We will assume in this section that $c > 0$, meaning the wind is going left to right. The upwind scheme is a Forward Euler method with a backward finite difference in space:

$$D_\tau \mathbf{U}_j^n + c\left(\frac{\mathbf{U}_j^{n-1} - \mathbf{U}_{j-1}^{n-1}}{h}\right) = \mathbf{f}_j^{n-1}.$$

Essentially, this scheme "looks upwind" to compute the solution to the transport equation. The figure below shows the upwind scheme

The truncation error of this scheme will be first order, due to the use of backward differences.

Proposition 3.1 (consistency of upwind scheme)**.** Let $\mathbf{u}_j^n = u(t_n, x_j)$, where u is an exact solution of (1). Then $\mathbf{u}_j^n$ solves the discrete equation

$$D_\tau \mathbf{u}_j^n + c\left(\frac{\mathbf{u}_j^{n-1} - \mathbf{u}_{j-1}^{n-1}}{h}\right) = \mathbf{f}^{n-1} + \boldsymbol{\tau}_{h,\tau}^{n-1},$$

where $\boldsymbol{\tau}_{h,\tau}^{n-1}$ is a truncation error and satisfies

$$\|\boldsymbol{\tau}_{h,\tau}^{n-1}\|_\infty \le C\left(h|u_{xx}|_{max} + \tau|u_{tt}|_{max}\right)$$

Proof. Use Taylor expansion at (t_{n-1}, x_j) similar to Forward Euler for the heat equation. $\square$

A more delicate issue is stability of the upwind scheme, which holds under a CFL condition.

Proposition 3.2 (stability of upwind scheme)**.** Let $\mathbf{U}_j^n$ be the discrete solution of the upwind scheme. Further assume that the following CFL condition holds

$$\tau \leq \frac{h}{c}.$$

Then,

$$\|\mathbf{U}^n\|_\infty \leq \|\mathbf{U}^{n-1}\|_\infty + \tau\|\mathbf{f}^{n-1}\|_\infty.$$

Proof. The proof follows similarly to that of stability of Forward Euler for the heat equation under a CFL condition.

We first suppose that we have j such that $\mathbf{U}_j^n = \max_{k\in\mathbb{Z}} |\mathbf{U}_k^n| = \|\mathbf{U}^n\|_\infty$. If there is no such positive $\mathbf{U}_j^n$, then we look at $-\mathbf{U}_j^n$. We now write the upwind iteration:

$$\mathbf{U}_j^n = \left(1 - \frac{c\tau}{h}\right)\mathbf{U}_j^{n-1} + \frac{c\tau}{h}\mathbf{U}_{j-1}^{n-1} + \tau\mathbf{f}_j^{n-1}$$

Since we are "upwinding", we can see that $\frac{c\tau}{h} \geq 0$, so

$$\frac{c\tau}{h}\mathbf{U}_{j-1}^{n-1} \leq \frac{c\tau}{h}\|\mathbf{U}^{n-1}\|_\infty.$$

Also, the CFL condition tells us $\left(1 - \frac{c\tau}{h}\right) \geq 0$, so

$$\left(1 - \frac{c\tau}{h}\right)\mathbf{U}_j^{n-1} \leq \left(1 - \frac{c\tau}{h}\right)\|\mathbf{U}^{n-1}\|_\infty.$$

We combine these estimates to get

$$\|\mathbf{U}^n\|_\infty = \mathbf{U}_j^n \leq \|\mathbf{U}^{n-1}\|_\infty + \tau\|\mathbf{f}^{n-1}\|_\infty,$$

which is the desired result. $\square$

Note that we can also sum the stability from $n = 1, \dots, N$ to get

$$\|\mathbf{U}^n\|_\infty \leq \|\mathbf{U}^0\|_\infty + \tau\sum_{n=1}^{N}\|\mathbf{f}^{n-1}\|_\infty,$$

which is a more useful stability result for the error estimate below.

Proposition 3.3 (error estimate of upwind scheme)**.** Let $\mathbf{U}_j^n$ be the discrete solution of the upwind scheme. Further assume that the following CFL condition holds

$$\tau \leq \frac{h}{c}.$$

Then, if $\mathbf{u}_j^n = u(t_n, x_j)$, we have the error estimate:

$$\|\mathbf{u}^n - \mathbf{U}^n\|_\infty \leq t_n C\left(h|u_{xx}|_{max} + \tau|u_{tt}|_{max}\right).$$

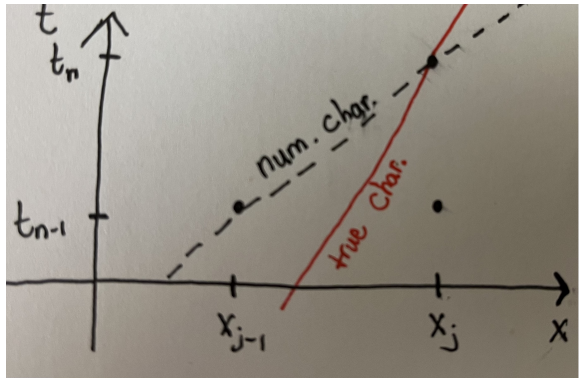

FIGURE 2. Red is the true characteristic and the black dashed line is the "numerical characteristic." The CFL condition states essentially that the numerical characteristic needs to move faster than the true characteristic.

Proof. Use typical techniques we learned from heat equation. Write down a discrete equation for the error $\mathbf{u}^n - \mathbf{U}^n$. Combine the stability estimate and the estimate on truncation error to complete the result. $\square$

Remark 3.1 (CFL condition for transport)**.** Note that the proof required the following CFL condition:

$$\tau \leq \frac{h}{c}.$$

I claim that this is a mild condition and is the best we can do for transport problems. The reason is that the fundamental property of transport equations are characteristics. Given a point (t_j, x_i), the numerical method needs to use enough points at t_{j-1} to capture the true characteristic. Note that the characteristic line (t_j, x_i) goes through t_{j-1} at $x = x_i - \tau c$. Thus, in order for $x \in (x_{i-1}, x_{i+1})$, we require that $\tau \leq \frac{h}{c}$.

One benefit of knowing that the CFL condition $\tau \leq \frac{h}{c}$ is the best we can expect means we can get away with a Forward Euler (hence cheaper to compute) method for transport. For the heat equation, we had a tradeoff between Backward Euler (expensive to compute) and Forward Euler (cheap to compute but bad CFL condition).