COMPUTATIONAL PDE LECTURES 20 AND 21

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1. Outline of these lectures

- Derivation of transport equation
- Method of characteristics

2. Setup

We are trying to solve the transport equation on the whole real line:

$$\begin{cases} u_t(t,x) + cu_x(t,x) = 0, \quad t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

where $c \neq 0$ is some constant speed.

3. Derivation of the Transport Equation

Suppose that u represents a mass density with units [kg/m] of some particles blowing in the wind where the wind velocity is c. A physical property of this system is **conservation of mass**, i.e. for

$$M(t) = \int_{ct}^{b+ct} u(t,x)dx$$

we have

$$\frac{d}{dt}M(t) = 0.$$

We can rewrite conservation of mass to derive the transport equation (1). Computing the time derivative, we use Leibniz integral rule:

$$\begin{aligned} \frac{d}{dt}M(t) &= \int_{ct}^{b+ct} u_t(t,x)dx + u(b+ct,x)\frac{d}{dt}(b+ct) - u(ct,x)\frac{d}{dt}(ct) \\ &= \int_{ct}^{b+ct} u_t(t,x)dx + c\big(u(b+ct,x) - u(ct,x)\big) \end{aligned}$$

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For these derivations, we typically want to compare apples to apples, i.e. we want everything as an integral over the same domain. Hence, we use Fundamental Theorem of Calculus to write:

$$u(b+ct,x) - u(ct,x) = \int_{ct}^{b+ct} u_x(t,x)dx,$$

which results in

$$0 = \int_{ct}^{b+ct} u_t(t,x) + cu_x(t,x)dx.$$

Dividing the above equation by b and taking a limit as $b \to$ shows that

$$0 = \lim_{b\to 0} \frac{1}{b}\int_{ct}^{b+ct} u_t(t,x) + cu_x(t,x)dx. = u_t(ct,x) + cu_x(ct,x).$$

Notice that ct is arbitrary so we have for any t, x:

$$u_t(t,x) + cu_x(t,x) = 0,$$

which is the desired PDE (1).

Remark 3.1 (other conservation laws)**.** The transport equation bears resemblance to many conservation laws of the form:

$$\frac{d}{dt}\int_a^b u(t,x)dx = -f(u(b)) + f(u(a)),$$

where f is the flux of some quantity u that is to be conserved. By applying Fundamental Theorem of Calculus to the RHS of the above equation and repeating arguments we used for the transport equation, we have

$$u_t(t,x) + \partial_x\big(f(u(t,x))\big) = 0,$$

which is referred to as a **nonlinear conservation law**. One such famous example is Burger's equation:

$$u_t + \partial_x\left(\frac{1}{2}u^2\right) = 0,$$

which is a common equation in fluid dynamics.

4. Solving the transport equation: method of characteristics

We are interested in solving (1):

$$\begin{cases} u_t(t,x) + cu_x(t,x) = 0, & t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases}.$$

We first begin with a geometric idea. Note that the RHS of (1) reads

$$0 = u_t(t,x) + cu_x(t,x) = \begin{pmatrix} 1 \\ c \end{pmatrix} \cdot \begin{pmatrix} u_t \\ u_x \end{pmatrix}.$$

Notice that the directional derivative of u in the direction $(1,c)$ is zero. Hence u must be constant along a line of the form $\{(t,x) : x = x_0 + ct\}$. This line is known as a **characteristic**.

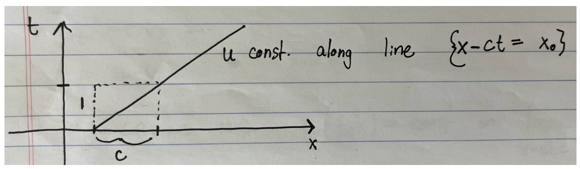

FIGURE 1. Characteristic lines where u is constant along the line.

Importantly, we then have that for any t:

$$u(t, x_0 + ct) = u(0, x_0) = u_0(x_0).$$

To then get the solution formula for u, we write $x_0 = x - ct$ to get

$$u(t,x) = u_0(x - ct). \tag{2}$$

We summarize this technique in the following proposition.

Proposition 4.1 (method of characteristics solution)**.** Let $u_0 \in C^1(\mathbb{R})$. Then u defined in (2) solves (1).

Proof. Clearly $u(0,x) = u_0(x)$. To check the PDE, we use chain rule:

$$\partial_t u(t,x) + c\partial_x u(t,x) = \partial_t[u_0(x-ct)] + c\partial_x[u_0(x-ct)] = -cu_0'(x-ct) + cu_0'(x-ct) = 0.$$

□

Remark 4.1 (physical interpretation of characteristics)**.** If a particle starts at x_0, then the line $\{(t,x) : x_0 + ct = x\}$ is the physical path that a particle takes in space.

Method of characteristics also shows us uniqueness.

Proposition 4.2 (method of characteristics solution)**.** Let $u_0 \in C^1(\mathbb{R})$. The solution to (1) is unique.

Proof. Let u, v be solutions to (1). Then their difference $w = u - v$ solves

$$\begin{cases} w_t(t,x) + cw_x(t,x) = 0, & t > 0, x \in \mathbb{R} \\ w(0,x) = 0 \end{cases}.$$

Recall that we know that w is constant along lines $x_0 + ct$. Hence,

$$w(t, x) = w(0, x - ct) = 0,$$

which completes the proof. □

Since we have an explicit solution formula and know that the solution is unique, we also have stability.

Proposition 4.3 (stability of transport)**.** Let $u_0 \in C^1(\mathbb{R})$ such that $\max_{z\in\mathbb{R}} |u_0(z)|$ is well-defined. The solution u to (1) satisfies for all t, x:

$$|u(t, x)| \leq \max_{z\in\mathbb{R}} |u_0(z)|$$

Proof. We use the solution formula in (2): $|u(t, x)| = |u_0(x - ct)| \leq \max_{z\in\mathbb{R}} |u_0(z)|$. □

4.1. **Method of characteristics with forcing.** We now consider a small generalization of (1) to

$$\begin{cases} u_t(t, x) + cu_x(t, x) = f(t, x), & t > 0, x \in \mathbb{R} \\ u(0, x) = u_0(x) \end{cases} \tag{3}$$

The motivation for this addition is that when we solve with finite differences, we introduce a small truncation error that will behave like f. To solve (3), we again look at the solution along characteristics. We define

$$v(t) = u(t, x_0 + ct),$$

which is u evaluated along a characteristic. Computing the time derivative of v with chain rule yields

$$v'(t) = u_t(t, x_0 + ct) + cu_x(t, x_0 + ct) = f(t, x_0 + ct),$$

and evaluating v at 0 gives

$$v(0) = u(0, x_0) = u_0(x_0).$$

To solve this ODE initial value problem, we only need to apply Fundamental Theorem of Calculus:

$$v(t) = v(0) + \int_0^t v'(s)ds = u_0(x_0) + \int_0^t f(s, x_0 + cs)ds.$$

We then use $x = x_0 + ct$ or $x - ct = x_0$ to get

$$u(t, x) = v(t) = u_0(x - ct) + \int_0^t f(s, x_0 + c(s - t))ds. \tag{4}$$

which solves (3). One can apply the arguments from the previous section to say that (3) admits a unique solution. Hence, we can use the solution (4) to state the following stability result.

Proposition 4.4 (stability of transport with forcing)**.** Let $u_0 \in C^1(\mathbb{R})$ such that $\max_{z\in\mathbb{R}} |u_0(z)|$ is well-defined. Let $f \in C^1((0,\infty)\times\mathbb{R})$ be such that $\max_{s>0,z\in\mathbb{R}} |f(s,z)|$ is well-defined. Let u be a solution to (3). Then u satisfies

$$|u(t,x)| \leq \max_{z\in\mathbb{R}} |u_0(z)| + t \max_{s>0,z\in\mathbb{R}} |f(s,z)|$$

Proof. We can again use an explicit solution formula (4):

$$u(t,x) = u_0(x-ct) + \int_0^t f(s, x_0 + c(s-t))ds.$$

We then take the absolute value and use triangle inequality

$$\begin{aligned} |u(t,x)| &\leq |u_0(x-ct)| + \left|\int_0^t f(s,x_0+c(s-t))ds\right| \\ &\leq \max_{z\in\mathbb{R}} |u_0(z)| + \left|\int_0^t f(s,x_0+c(s-t))ds\right| \\ &\leq \max_{z\in\mathbb{R}} |u_0(z)| + \int_0^t \big|f(s,x_0+c(s-t))\big|\, ds \\ &\leq \max_{z\in\mathbb{R}} |u_0(z)| + t \max_{s>0,z\in\mathbb{R}} |f(s,z)| \end{aligned}$$

which is the desired result. □

Note that this stability formula will be more useful in the analysis of a numerical method since the numerical method will have a small f, that is a truncation error. Our finite difference scheme will try to mimic this property.

4.2. **Method of characteristics for more general situations.** Lots of things in the worlds do not have constant velocity, so it makes sense to model systems where the velocity of a particle at a point is $c(t,x)$ and depends on space and time. In this case, we can following the physical derivation at the beginning of these notes to find a new transport equation:

$$u_t(t,x) + c(t,x)u_x(t,x) + c_x(t,x)u(t,x) = 0.$$

This particular equation now looks quite different from the transport equation with constant velocity. However, the method of characteristics still works as long as we remember that we an easily solve for u along the path a particle takes.

4.2.1. *General procedure.* We consider the following equation

$$\begin{cases} u_t(t,x) + c(t,x)u_x(t,x) + b(t,x)u(t,x) = f(t,x), & t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases} \tag{5}$$

We break the solution into two steps:
Step 1: Find the characteristics We first search for the characteristics of (5). With the interpretation that this the path a particle takes with velocity $c(t,x)$, we find a solution to the ODE:

$$\begin{cases} y'(t) = c(t, y(t)), t > 0 \\ y(0) = x_0 \end{cases}.$$

The solution y describes the position of a particle that started at position x_0 and evolved with velocity $c(t, y(t))$.

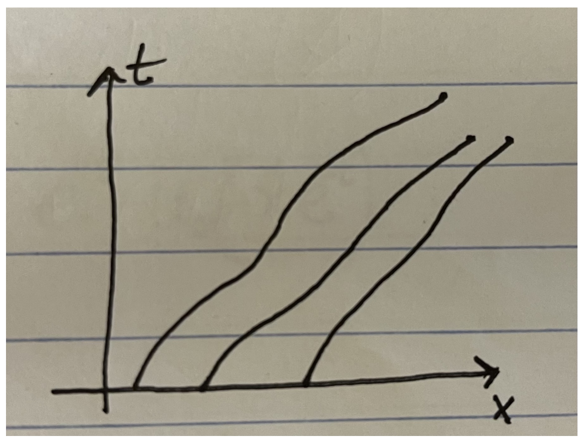

FIGURE 2. Characteristics $\{(t, y(t)) : t > 0\}$ that solve $y'(t) = c(t, y(t))$

Step 2: Solve for u along characteristics Similar to the case with forcing, we solve for u along characteristics. That is, we define $v(t) = u(t, y(t))$, and find the ODE for v. We have from chain rule and the PDE:

$$v'(t) = u_t(t,y(t)) + u_x(t,y(t))y'(t) = -b(t,y(t))u(t,y(t)) + f(t,y(t)) = -b(t,y(t))v(t) + f(t,y(t)).$$

We also have the initial condition

$$v(0) = u(0, x_0) = u_0(x_0).$$

Combining these leads to solving the initial value problem:

$$\begin{cases} v'(t) = -b(t, y(t))v(t) + f(t, y(t)), t > 0 \\ v(0) = u_0(x_0) \end{cases}.$$

We complete the construction by writing x_0 in terms of $x = y(t)$ and t.

4.2.2. *Examples.* We now look at 2 examples.

Example 4.1. Consider $c(t, x) = x, f(t, x) = 0, b(t, x) = 1$, so

$$\begin{cases} u_t(t, x) + xu_x(t, x) + u = 0, \quad t > 0, x \in \mathbb{R} \\ u(0, x) = u_0(x) \end{cases}$$

The ODE for characteristic in Step 1 is

$$\begin{cases} y'(t) = y(t), t > 0 \\ y(0) = x_0 \end{cases}$$

whose solution is $y(t) = x_0e^t$.

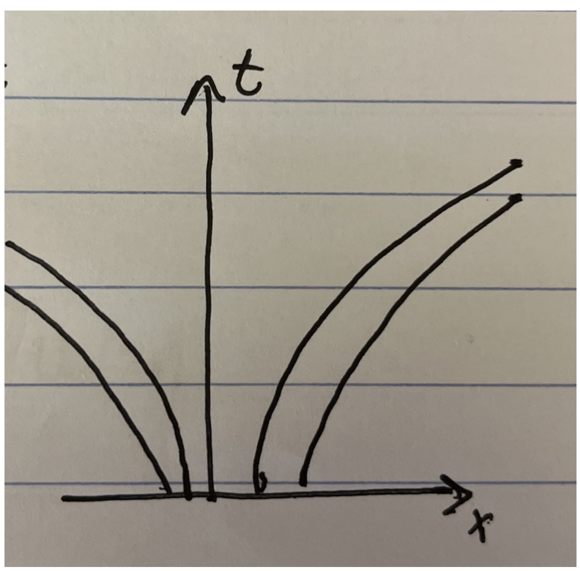

FIGURE 3. Characteristics $y(t) = x_0e^t$

The ODE for the solution $u(t, y(t)) = v(t)$ along a characteristic in Step 2 is

$$\begin{cases} v'(t) = -v(t), t > 0 \\ v(0) = u(x_0) \end{cases}$$

so

$$u(t, y(t)) = v(t) = e^{-t} u_0(x_0).$$

Setting $x = y(t)$, we have $x_0 = xe^{-t}$, and the solution is

$$u(t, x) = e^{-t} u_0(xe^{-t}).$$

Example 4.2. Consider $c(t, x) = t, b(t, x) = 0$, and $f(t, x) = 1$. The transport equation reads:

$$\begin{cases} u_t(t, x) + t u_x(t, x) = 1, \quad t > 0, x \in \mathbb{R} \\ u(0, x) = u_0(x) \end{cases}$$

The ODE for characteristic in Step 1 is

$$\begin{cases} y'(t) = t, t > 0 \\ y(0) = x_0 \end{cases}$$

whose solution is $y(t) = x_0 + \frac{t^2}{2}$.

The ODE for the solution $u(t, y(t)) = v(t)$ along a characteristic in Step 2 is

$$\begin{cases} v'(t) = 1, t > 0 \\ v(0) = u(x_0) \end{cases}$$

so

$$u(t, y(t)) = v(t) = 1 + u_0(x_0).$$

Writing $x = y(t) = x_0 + \frac{t^2}{2}$, we have the solution

$$u(t, x) = 1 + u_0 \left(x - \frac{t^2}{2} \right)$$