COMPUTATIONAL PDE LECTURE 15

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1. Outline of today

- Prove Error Estimate for Backward Euler
- Prove alternative stability estimates for Backward/Forward Euler

2. Finite Difference Methods for the Heat Equation

We are trying to solve:

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = f(t,x), \quad t \in (0,1), x \in (0,1) \\ u(t,0) = 0, u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

We now list with various time stepping schemes for the heat equation, which you implemented in recitation:

- **Forward Euler** (approximates differential equation at t_{j-1})

$$D_\tau \mathbf{U}^j + \mathbf{A}^h \mathbf{U}^{j-1} = \mathbf{f}^{j-1}$$

- **Backward Euler** (approximates differential equation at t_j)

$$D_\tau \mathbf{U}^j + \mathbf{A}^h \mathbf{U}^j = \mathbf{f}^j$$

2.1. **Error estimate for Backward Euler.**

Recall that we have shown consistency of both schemes.

Proposition 2.1 (consistency of schemes)**.** Let u solve (1). Then $\mathbf{u}_i^j = u(t_j, x_i)$ satisfies

- **Forward Euler**

$$D_\tau \mathbf{u}^j + \mathbf{A}^h \mathbf{u}^{j-1} = \mathbf{f}^{j-1} + \boldsymbol{\tau}^j_{\tau,h,fe}$$

- **Backward Euler**

$$D_\tau \mathbf{u}^j + \mathbf{A}^h \mathbf{u}^j = \mathbf{f}^j + \boldsymbol{\tau}^j_{\tau,h,be}$$

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where there is a constant $C > 0$ such that the truncation errors satisfy:

$$\|\boldsymbol{\tau}^j_{\tau,h,fe}\|_\infty \leq C\left(\tau|u_{tt}|_{max} + h^2|u_{xxxx}|_{max}\right)$$
$$\|\boldsymbol{\tau}^j_{\tau,h,be}\|_\infty \leq C\left(\tau|u_{tt}|_{max} + h^2|u_{xxxx}|_{max}\right)$$

and

$$|u_{tt}|_{max} = \max_{t,x\in[0,1]}|u_{tt}(t,x)|, \quad |u_{ttt}|_{max} = \max_{t,x\in[0,1]}|u_{ttt}(t,x)|, \quad |u_{xxxx}|_{max} = \max_{t,x\in[0,1]}|u_{xxxx}(t,x)|$$

We also have shown stability of Backward Euler.

Proposition 2.2 (unconditional stability of Backward Euler)**.** Let $\mathbf{U}^j$ be the sequence of solutions to the Backward Euler scheme:

$$D_\tau \mathbf{U}^j + \mathbf{A}^h\mathbf{U}^j = \mathbf{f}^j,$$

then the vector $\mathbf{U}^j$ satisfies the following discrete energy estimate:

$$\|\mathbf{U}^J\|^2_{2,h} + \sum_{j=1}^{J}\left[\tau\|\mathbf{U}^j\|^2_{\mathbf{A}^h} + \|\mathbf{U}^j - \mathbf{U}^{j-1}\|^2_{2,h}\right] \leq \|\mathbf{U}^0\|_{2,h} + \sum_{j=1}^{J}\tau\|\mathbf{f}^j\|^2_{2,h} \tag{2}$$

where

$$\|\mathbf{v}\|^2_{2,h} = \sum_{i=1}^{N-1} h\mathbf{v}_i^2, \quad \|\mathbf{v}\|^2_{\mathbf{A}^h} = h\mathbf{v}^T\mathbf{A}^h\mathbf{v}.$$

Recall that we have

$$\text{Stability} + \text{Consistency} \implies \text{Convergence}$$

We now combine the two results to show an error estimate.

Proposition 2.3 (error estimate of Backward Euler)**.** Let u solve (1). Let $\mathbf{U}^j$ be the sequence of solutions to the Backward Euler scheme:

$$D_\tau \mathbf{U}^j + \mathbf{A}^h\mathbf{U}^j = \mathbf{f}^j$$

Then, we define $\mathbf{e}^j = \mathbf{u}^j - \mathbf{U}^j$. Let $T = J\tau$. The error satisfies for some $C > 0$:

(3)

$$\|\mathbf{e}^J\|^2_{2,h} + \sum_{j=1}^{J}\tau\|\mathbf{e}^j\|^2_{\mathbf{A}^h} \leq \|\mathbf{e}^0\|_{2,h} + \sum_{j=1}^{J}\tau\|\boldsymbol{\tau}^j_{\tau,h,be}\|^2_{2,h} \leq CT\left(\tau^2|u_{ttt}|^2_{max} + h^4|u_{xxxx}|^2_{max}\right)$$

Moreover,

$$\|\mathbf{e}^J\|_{2,h} \leq \sqrt{CT}\left(\tau|u_{ttt}|_{max} + h^2|u_{xxxx}|_{max}\right)$$

Proof. Recall that

$$D_\tau \mathbf{u}^j + \mathbf{A}^h \mathbf{u}^j = \mathbf{f}^j + \boldsymbol{\tau}^j_{\tau,h,be},$$

and

$$D_\tau \mathbf{U}^j + \mathbf{A}^h \mathbf{U}^j = \mathbf{f}^j.$$

Subtracting these two equations leads to

$$D_\tau \mathbf{e}^j + \mathbf{A}^h \mathbf{e}^j = \boldsymbol{\tau}^j_{\tau,h,be}.$$

Applying the stability estimate (2) to $\mathbf{e}^j$ and the estimates on $\boldsymbol{\tau}^j_{\tau,h,be}$ give the desired result. □

2.2. **Alternative Stability Result.** Note that the stability result we used mimics the energy estimates of the heat equation. We can actually show an even better stability result by mimicking the maximum principle of the heat equation.

Proposition 2.4 (∞-norm stability of Euler)**.** Let $\mathbf{U}^j$ be the sequence of solutions to the Backward Euler scheme. Then the vector $\mathbf{U}^J$ satisfies:

$$\|\mathbf{U}^J\|_\infty \le \|\mathbf{U}^0\|_\infty + \sum_{j=1}^{J} \tau \|\mathbf{f}^j\|_\infty.$$

Let $\mathbf{U}^j$ be the sequence of solutions to the Backward Euler scheme. Then if

$$\tau \le \frac{h^2}{2} \tag{4}$$

the vector $\mathbf{U}^j$ satisfies:

$$\|\mathbf{U}^J\|_\infty \le \|\mathbf{U}^0\|_\infty + \sum_{j=0}^{J-1} \tau \|\mathbf{f}^j\|_\infty.$$

Proof of B.E. estimate. Suppose i is such that $\mathbf{U}^j_i = \|\mathbf{U}^j\|_\infty$. Then the Backward Euler iteration reads:

$$\text{LHS} = \left(1 + \frac{2\tau}{h^2}\right)\mathbf{U}^j_i - \frac{\tau}{h^2}\left(\mathbf{U}^j_{i-1} + \mathbf{U}^j_{i+1}\right) = \mathbf{U}^{j-1}_i + \tau \mathbf{f}^j_i \le \|\mathbf{U}^{j-1}\|_\infty + \tau\|\mathbf{f}^j_i\|_\infty.$$

The LHS can be estimated from below. This essentially uses the fact that $\mathbf{A}^h$ is an M matrix and $-\mathbf{U}^j_i = -\|\mathbf{U}^j\|_\infty \le -\mathbf{U}^j_{i\pm1}$:

$$\text{LHS} \ge \left(1 + \frac{2\tau}{h^2}\right)\mathbf{U}^j_i - \frac{2\tau}{h^2}\mathbf{U}^j_i = \mathbf{U}^j_i = \|\mathbf{U}^j\|_\infty.$$

We now have

$$\|\mathbf{U}^j\|_\infty \le \|\mathbf{U}^{j-1}\|_\infty + \tau\|\mathbf{f}^j_i\|_\infty.$$

If there is no i is such that $\mathbf{U}_i^j = \|\mathbf{U}^j\|_\infty$, we'd then have an i such that i is such that $-\mathbf{U}_i^j = \|\mathbf{U}^j\|_\infty$. Then repeat the argument with $-\mathbf{U}^j$. Then apply the inequality inductively to get the final result. □

Proof of F.E. estimate. We repeat the same argument as for B.E. Suppose $\mathbf{U}_i^j = \|\mathbf{U}^j\|_\infty$. The FE scheme reads

$$\mathbf{U}_i^j = \left(1 - \frac{2\tau}{h^2}\right)\mathbf{U}_i^{j-1} + \frac{\tau}{h^2}\left(\mathbf{U}_{i-1}^{j-1} + \mathbf{U}_{i+1}^{j-1}\right) + \tau\mathbf{f}_i^{j-1} = \text{ RHS.}$$

Since we assume

$$\tau \leq \frac{h^2}{2},$$

then

$$\left(1 - \frac{2\tau}{h^2}\right) \geq 0, \text{ and } \left(1 - \frac{2\tau}{h^2}\right)\mathbf{U}_i^{j-1} \leq \left(1 - \frac{2\tau}{h^2}\right)\|\mathbf{U}j - 1\|_\infty.$$

We can then estimate the RHS by

$$\text{RHS} \leq \left(1 - \frac{2\tau}{h^2}\right)\|\mathbf{U}^{j-1}\|_\infty + 2\frac{\tau}{h^2}\|\mathbf{U}^{j-1}\|_\infty + \tau\|\mathbf{f}_i^{j-1}\|_\infty = \|\mathbf{U}^{j-1}\|_\infty + \tau\|\mathbf{f}_i^{j-1}\|_\infty.$$

Hence,

$$\|\mathbf{U}^j\|_\infty \leq \|\mathbf{U}^{j-1}\|_\infty + \tau\|\mathbf{f}_i^{j-1}\|_\infty.$$

□

Remark 2.1 (CFL Condition)**.** For the stability of Forward Euler, we required

$$\tau \leq \frac{h^2}{2_{max}}. \tag{5}$$

This is known as a CFL (Courant-Friedrichs-Lewy) condition. Although Forward Euler is cheaper at each step than backward Euler, we pay with a restriction on the size of the time-step size.