COMPUTATIONAL PDE LECTURES 26-31

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1. Outline of these lectures

- Derive wave equation
- Solve the wave equation on the real line using a technique due to d'Alembert.
- Show energy estimates for the wave equation on the real line
- Discuss separation of variables for the wave equation.
- Explicit finite difference method for the wave equation: consistency and von Neumann analysis
- Derivation of implicit but energy conserving finite difference method for the wave equation

2. Setup

We will first look at the wave equation on the whole real line:

$$\begin{cases} u_{tt}(t,x) - c^2 u_{xx}(t,x) = 0, \quad t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x), \quad x \in \mathbb{R} \\ u_t(0,x) = g_0(x), \quad x \in \mathbb{R} \end{cases} \tag{1}$$

Note that we now have a new initial condition, which specifies the initial velocity $u_t(0,x) = g_0(x)$. This is because we now have two derivatives in time on u.

3. Derivation of wave equation

We first begin with a derivation of the wave equation (1). Here, u represents the height of a string with mass density ρ that has constant tension, T, throughout the string. We now look at a free-body diagram of the string

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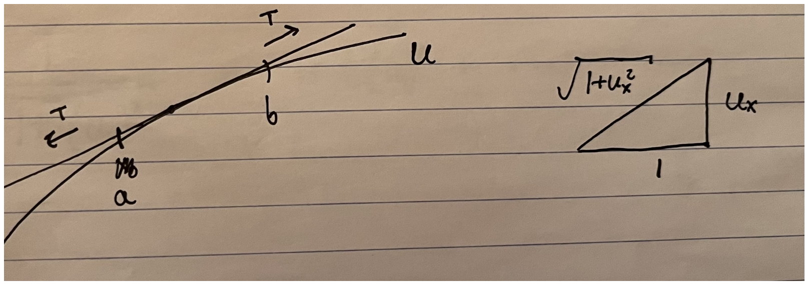

FIGURE 1. Free body diagram of string on left. The triangle on the right shows the decomposition of the forces into vertical and horizontal components.

Writing out Newton's second law $F = ma$, where F is the force, m is the mass, and a is the acceleration, we now break the forces into a vertical and horizontal component. The horizontal component is

$$\frac{T}{\sqrt{1+u_x(t,b)^2}} - \frac{T}{\sqrt{1+u_x(t,a)^2}} = 0.$$

The vertical component is

$$\frac{Tu_x(t,b)}{\sqrt{1+u_x(t,b)^2}} - \frac{Tu_x(t,a)}{\sqrt{1+u_x(t,a)^2}} = \int_a^b \rho u_{tt}(t,x)dx.$$

Note that the vertical component equation is of the form $F = ma$, since we have the vertical component of the tension force on the LHS, and the RHS is an integral of density times acceleration, which has units of mass times acceleration after an integral in space.

In order to compare apples to apples, we now rewrite the LHS in terms of an integral using Fundamental Theorem of Calculus and we have

$$\int_a^b \partial_x \left[\frac{Tu_x(t,x)}{\sqrt{1+u_x(t,x)^2}}\right] dx = \int_a^b \rho u_{tt}(t,x)dx.$$

We now use the same usual trick by dividing by $b-a$ and taking a limit as $b \to a$ to get

$$\rho u_{tt}(t,x) - \partial_x \left[\frac{Tu_x(t,x)}{\sqrt{1+u_x(t,x)^2}}\right] = 0$$

This equation is highly nonlinear and is difficult to study. To get the wave equation, we now make an assumption that u_x is small or $u_x \approx 0$. Then we have the Taylor expansion:

$$\frac{1}{\sqrt{1+u_x(t,x)^2}} = 1 - \frac{u_x(t,x)^2}{2} + \mathcal{O}(u_x^4),$$

so

$$\frac{Tu_x(t,x)}{\sqrt{1+u_x(t,x)^2}} = Tu_x(t,x) + \mathcal{O}(u_x^2).$$

We then apply the derivative in ∂_x to get

$$\partial_x\left[\frac{Tu_x(t,x)}{\sqrt{1+u_x(t,x)^2}}\right] \approx Tu_{xx}(t,x).$$

The resulting equation after making this smallness assumption is the wave equation in (1)

$$u_{tt}(t,x) - c^2 u_{xx}(t,x) = 0,$$

where $c = \sqrt{\frac{T}{\rho}}$. Note that c here has units of speed. We'll see later that waves travel at speed c.

Remark 3.1 (smallness assumption)**.** Recall that we had to make a small amplitude assumption, i.e. $u_x \approx 0$. This is a common modeling technique to go from a nonlinear equation to a linear equation. Often, many linear theories are approximations of the more correct nonlinear theory, where we assume some quantity is small. For instance, if we have a steady state solution of the nonlinear wave equation ($u_{tt} = 0$), then we have the equation

$$-\partial_x\left[\frac{u_x(t,x)}{\sqrt{1+u_x(t,x)^2}}\right] = 0,$$

which is the equation of a line minimizing arc length. The linearized theory (assuming $u_x \approx 0$) is

$$-u_{xx}(t,x) = 0,$$

which is Laplace's equation.

Another example if you ever take a course in solid mechanics is you'll most likely study linear elasticity. This theory is making assumptions that the deformation of the material is small. The richer set of theories that are nonlinear are able to describe large deformations better.

4. d'Alembert's solution of the wave equation

We now want to construct a solution to (1). The main idea due to d'Alembert was to factor the equation into:

$$0 = u_{tt}(t,x) - c^2 u_{xx}(t,x) = (\partial_{tt} - c^2\partial_{xx})u(t,x) = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u(t,x).$$

This factorization is motivated by $(a+b)(a-b) = a^2 - b^2$ for any two real numbers a, b. Hence, if we define $(\partial_t + c\partial_x)u(t,x) = w(t,x)$, then w solves a transport equation $(\partial_t - c\partial_x)w(t,x) = 0$. We then have the following system of transport equations:

$$\begin{cases} u_t + cu_x = w \\ w_t - cw_x = 0 \end{cases}.$$

We know how to solve transport equations using the method of characteristics. We have that u is

$$u(t,x) = u_0(x-ct) + \int_0^t w(s, x+c(s-t))ds,$$

and w is

$$w(t,x) = w_0(x+ct) = \partial_t u(0, x+ct) - c\partial_x u(0, x+ct) = g_0(x+ct) - cu_0'(x+ct).$$

We now substitute the solution for w into the integral in the solution formula for u in order to write everything in terms of the initial data. We have

$$w(s, x+c(s-t)) = w_0(x+c(s-t)+cs) = g_0(x+c(s-t)+cs) + cu_0'(x+c(s-t)+cs),$$

so

$$u(t,x) = u_0(x-ct) + \int_0^t g_0(x+c(s-t)+cs) + cu_0'(x+c(s-t)+cs)ds.$$

The integral terms we can simplify using a change of variables $y = x + c(s-t) + cs$, so then

$$u(t,x) = u_0(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_0(y) + cu_0'(y)dy.$$

The last term can be further simplified using Fundamental Theorem of Calculus

$$\frac{1}{2}\int_{x-ct}^{x+ct} u_0'(y)dy = \frac{1}{2}\left(u_0(x+ct) - u_0(x-ct)\right).$$

Hence,

$$u(t,x) = \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g_0(y)dy, \tag{2}$$

which is the d'Alembert solution to the wave equation (1). This technique shows two important properties of the wave equation.

Example 4.1 (plucked string)**.** We now look at a simple example, where

$$u_0(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

and $g_0(x) = 0$. This is if we pull a string up with height 1. The solution is then

$$u(t,x) = \frac{1 - |x + ct|}{2} + \frac{1 - |x - ct|}{2},$$

which are two triangular waves of height $\frac{1}{2}$ traveling with velocities $-c$ and $+c$.

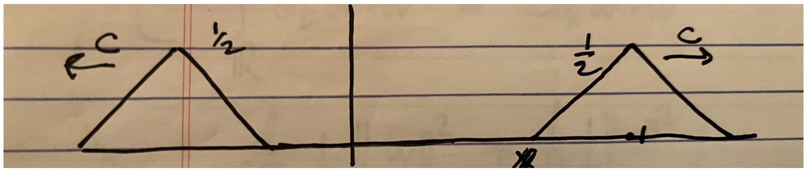

FIGURE 2. Solution to wave equation of plucked string.

Remark 4.1 (domain of dependence)**.** Let (t^*, x^*) be a point in space-time. We then have that $u(t^*, x^*)$ depends on the initial values $(t = 0)$ in the interval $[t^* - cx^*, t^* + cx^*]$. More generally, we can modify the d'Alembert formula in (2) for any $t \leq t^*$:

$$u(t^*, x^*) = \frac{1}{2}\left[u(t^*, x^* + c(t^* - t)) + u_0(t^*, x^* - c(t^* - t))\right] + \frac{1}{2c}\int_{x^* - c(t^* - t)}^{x^* + c(t^* - t)} u_t(t, x)dx.$$

This change in the formula shows that the solution $u(t^*, x^*)$ also depends on information of u on the set $\{t\} \times [x^* - c(t^* - t), x^* + c(t^* - t)]$. Looking at all $0 \leq t \leq t^*$, we have that $u(t^*, x^*)$ depends on u in the cone

$$C_{t^*,x^*} = \{(t, x) : x^* - c(t^* - t) \leq x \leq x^* + c(t^* - t)\}.$$

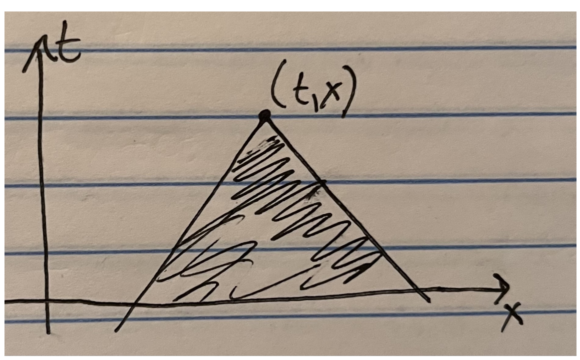

FIGURE 3. Domain of dependence of wave equation in shaded region.

This cone is known as the **domain of dependence**. We can also see that information travels at speed c. This will suggest a CFL condition $\tau \leq h/c$ for the numerics.

Remark 4.2 (coupled transport equations)**.** The factoring of the operator $(\partial_{tt} - c^2\partial_{xx}) = (\partial_t - c\partial_x)(\partial_t + c\partial_x)$ led to us solving a system of coupled transport equations:

$$\begin{cases} u_t + cu_x = w \\ w_t - cw_x = 0 \end{cases}.$$

The above set of equations suggests that we can might be able to use our techniques from the transport equation to solve the wave equation. We'll also use the above structure to derive energy estimates and show uniqueness of solutions to the wave equation.

5. Energy estimates for the wave equation and uniqueness

This section derives energy estimates for the wave equation and will consequently show uniqueness. In order to have a result that will translate to numerics more easily. We consider the wave equation with forcing.

$$\begin{cases} u_{tt}(t,x) - c^2 u_{xx}(t,x) = f(t,x), & t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x), & x \in \mathbb{R} \\ u_t(0,x) = g_0(x), & x \in \mathbb{R} \end{cases} \tag{3}$$

We now derive energy estimates for the wave equation.

Proposition 5.1 (first energy estimate)**.** Let u be a C^2 solution to (3) with fast decay at $\pm\infty$. Then,

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}} u(t,x)^2 + (u_t(t,x) + cu_x(t,x))^2 dx \\ &\leq e^{2t}\left(\frac{1}{2}\int_{\mathbb{R}} u(0,x)^2 + (u_t(0,x) + cu_x(0,x))^2 dx + \int_0^t \int_{\mathbb{R}} f(s,x)^2 dx ds\right). \end{aligned}$$

Proof. We write $w = u_t + cu_x$ and write the system of coupled transport equations:

$$\begin{aligned} u_t + cu_x &= w \\ w_t - cw_x &= f. \end{aligned}$$

We then multiply the equations by u and w respectively and integrate over $\mathbb{R}$ to get

$$\int_{\mathbb{R}} u_t(t,x)u(t,x) + cu_x(t,x)u(t,x)dx = \int_{\mathbb{R}} w(t,x)u(t,x)dx$$
$$\int_{\mathbb{R}} w_t(t,x)w(t,x) - cw_x(t,x)w(t,x) = \int_{\mathbb{R}} f(t,x)w(t,x)dx.$$

We now deal with the first equation. We first have by chain rule and Leibniz rule

$$\int_{\mathbb{R}} u_t(t,x)u(t,x)dx = \frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} u(t,x)^2 dx.$$

The second term actually integrates to zero due to the fast decay at ∞ assumption:

$$\int_{\mathbb{R}} u_x(t,x)u(t,x)dx = \lim_{L\to\infty}\int_{-L}^{L} u_x(t,x)u(t,x)dx = \lim_{L\to\infty}\int_{-L}^{L} \partial_x \frac{1}{2}u(t,x)^2 dx$$
$$= \lim_{L\to\infty}\left(u(t,L)^2 - u(t,-L)^2\right) = 0.$$

Finally the RHS can be handled with Young's inequality:

$$\int_{\mathbb{R}} w(t,x)u(t,x)dx \le \frac{1}{2}\int_{\mathbb{R}} w(t,x)^2 + u(t,x)^2 dx.$$

This results in the estimate:

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} u(t,x)^2 dx \le \frac{1}{2}\int_{\mathbb{R}} w(t,x)^2 + u(t,x)^2 dx.$$

We can apply the same techniques to the equation for w to get

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} u(t,x)^2 dx \le \frac{1}{2}\int_{\mathbb{R}} w(t,x)^2 + f(t,x)^2 dx.$$

Adding these two estimates together yields the inequality

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} u(t,x)^2 + w(t,x)^2 dx \le \int_{\mathbb{R}} w(t,x)^2 + u(t,x)^2 dx + \frac{1}{2}\int_{\mathbb{R}} f(t,x)^2 dx.$$

Notice that the LHS on the energy estimate we want to prove is $\int_{\mathbb{R}} u(t,x)^2 + w(t,x)^2 dx$. Let $\phi(t) = \int_{\mathbb{R}} u(t,x)^2 + w(t,x)^2 dx$. We currently have the inequality

$$\phi'(t) \le 2\phi(t) + \frac{1}{2}\int_{\mathbb{R}} f(t,x)^2 dx.$$

The desired energy estimate

$$\phi(t) \le e^{2t}\left(\phi(0) + \int_0^t \int_{\mathbb{R}} f(s,x)^2 dx ds\right)$$

is a consequence of Gronwall's inequality, which we prove below. □

Lemma 5.1 (Gronwall inequality)**.** Suppose $\phi \in C^1$ satisfies

$$\phi'(t) \leq \beta(t) + \alpha\phi(t),$$

where $\beta(t) \geq 0$ for all t and $\alpha \geq 0$. Then,

$$\phi(t) \leq e^{\alpha t}\left(\int_0^t \beta(s)ds + \phi(0)\right).$$

Proof. If ϕ satisfies the above inequality, then ϕ solves the ODE:

$$\phi'(t) = \gamma(t) + \alpha\phi(t),$$

with $\gamma(t) \leq \beta(t)$. The solution to the above ODE is

$$\phi(t) = e^{\alpha t}\phi(0) + \int_0^t e^{\alpha(t-s)}\gamma(s)ds.$$

We can then bound the second term using the fact that $e^{\alpha(t-s)} \leq e^{\alpha t}$ for all $0 \leq s \leq t$ and $\gamma(s) \leq \beta(s)$ for all s:

$$\phi(t) = e^{\alpha t}\phi(0) + \int_0^t e^{\alpha(t-s)}\gamma(s)ds \leq e^{\alpha t}\left(\phi(0) + \int_0^t \beta(s)ds\right),$$

which completes the proof. □

An important consequence of the energy estimate is that solutions that decay quickly at ∞ to the wave equation are unique.

Proposition 5.2 (uniqueness of solution to wave equation)**.** A C^2 solution to (3) with fast decay at ∞ is unique.

Proof. Let $u, v \in C^2$ be two such solutions to the wave equation (3). Notice that their difference $e = u - v$ solves

$$\begin{cases} e_{tt}(t,x) - c^2 e_{xx}(t,x) = 0, & t > 0, x \in \mathbb{R} \\ e(0,x) = 0, & x \in \mathbb{R} \\ e_t(0,x) = 0, & x \in \mathbb{R} \end{cases}.$$

We apply the energy estimate to get

$$\int_{\mathbb{R}} e(t,x)^2 + (e_t(t,x) + ce_x(t,x))^2 dx \leq 0.$$

Importantly $\int_{\mathbb{R}} e(t,x)^2 dx = 0$, so $e(t,x) = 0$ for all t, x. □

6. Solving the wave equation on an interval: separation of variables

We now depart from solving the wave equation on $\mathbb{R}$ and look at the interval $(0, 1)$. We are interested in solving

$$\begin{cases} u_{tt}(t,x) - c^2 u_{xx}(t,x) = 0, & t > 0, x \in (0,1) \\ u(t,0) = u(t,1) = 0, & t > 0 \\ u(0,x) = u_0(x), & x \in (0,1) \\ u_t(0,x) = g_0(x), & x \in (0,1) \end{cases} . \tag{4}$$

The boundary conditions in this case are homogenous Dirichlet boundary conditions. We now discuss separation of variables to solve (4). It follows very similarly to the heat equation.

For Dirichlet boundary conditions, like the heat equation, we seek solutions of the form:

$$u(t,x) = \sum_{k=1}^{\infty} T_k(t) \sin(k\pi x),$$

where T_k is an unknown function of time. We also compute Fourier sine series for u_0 and g_0.

$$\begin{aligned} u_0(x) &= \sum_{k=1}^{\infty} a_k \sin(k\pi x) \\ g_0(x) &= \sum_{k=1}^{\infty} b_k \sin(k\pi x). \end{aligned}$$

We now plug u into the PDE to see that

$$\begin{aligned} \sum_{k=1}^{\infty} (T_k''(t) + (c^2k^2\pi^2) T_k(t)) \sin(k\pi x) &= 0 \\ \sum_{k=1}^{\infty} T_k(0) \sin(k\pi x) &= \sum_{k=1}^{\infty} a_k \sin(k\pi x) \\ \sum_{k=1}^{\infty} T_k'(0) \sin(k\pi x) &= \sum_{k=1}^{\infty} b_k \sin(k\pi x). \end{aligned}$$

Just like the heat equation, we can now solve for each T_k by solving the following initial value problem:

$$\begin{aligned} T_k''(t) + (c^2k^2\pi^2)T_k(t) &= 0 \\ T_k(0) &= a_k \\ T_k'(0) &= b_k. \end{aligned}$$

Using techniques from ODEs, we can find the solution to this IVP as

$$T_k(t) = a_k \cos(ck\pi t) + \frac{b_k}{ck\pi}\sin(ck\pi t),$$

and the solution to (4) is

$$u(t,x) = \sum_{k=1}^{\infty}\left(a_k \cos(ck\pi t) + \frac{b_k}{ck\pi}\sin(ck\pi t)\right)\sin(k\pi x)$$

We now go through some examples that are modifications of (4).

Example 6.1 (string with springs)**.** Suppose the string we pluck has springs attached to it. The resulting PDE would be

$$\begin{cases} u_{tt}(t,x) - c^2u_{xx}(t,x) + Ku(t,x) = 0, & t>0, x\in(0,1) \\ u(t,0) = u(t,1) = 0, & t>0 \\ u(0,x) = u_0(x), & x\in(0,1) \\ u_t(0,x) = g_0(x), & x\in(0,1) \end{cases}, \tag{5}$$

where $K \geq 0$ is a spring constant. We again suppose u takes the form

$$u(t,x) = \sum_{k=1}^{\infty} T_k(t)\sin(k\pi x).$$

Plugging u into (5) leads to the following IVP for teach T_k:

$$\begin{aligned} T_k''(t) + (c^2k^2\pi^2 + K)T_k(t) &= 0 \\ T_k(0) &= a_k \\ T_k'(0) &= b_k. \end{aligned}$$

Using techniques from ODEs, we can find the solution to this IVP as

$$T_k(t) = a_k\cos(\sqrt{c^2k^2\pi^2+K}t) + \frac{b_k}{\sqrt{c^2k^2\pi^2+K}}\sin(\sqrt{c^2k^2\pi^2+K}t),$$

and the solution to (5) is

$$u(t,x) = \sum_{k=1}^{\infty}\left(a_k\cos(\sqrt{c^2k^2\pi^2+K}\ t) + \frac{b_k}{\sqrt{c^2k^2\pi^2+K}}\sin(\sqrt{c^2k^2\pi^2+K}\ t)\right)\sin(k\pi x).$$

Example 6.2 (string with springs and friction)**.** We slightly modify the last example and now add friction to the system.

$$(6)\qquad \begin{cases} u_{tt}(t,x) + \mu u_t(t,x) - c^2 u_{xx}(t,x) + Ku(t,x) = 0, & t > 0, x \in (0,1) \\ u(t,0) = u(t,1) = 0, & t > 0 \\ u(0,x) = u_0(x), & x \in (0,1) \\ u_t(0,x) = g_0(x), & x \in (0,1) \end{cases},$$

where $\mu \geq 0$ is a friction coefficient.

Plugging u into (6) leads to the following IVP for teach T_k:

$$\begin{aligned} T_k''(t) + \mu T_k'(t) + (c^2k^2\pi^2 + K)T_k(t) = 0 \\ T_k(0) = a_k \\ T_k'(0) = b_k. \end{aligned}$$

Using techniques from ODEs, we can find the solution to this IVP (assuming we are underdampled: $c^2k^2\pi^2 + K - \mu/2 \geq 0$):

$$T_k(t) = e^{-\mu t/2} c_k \cos(\sqrt{c^2k^2\pi^2 + K - \mu/2}\; t) + \tilde{c_k} \sin(\sqrt{c^2k^2\pi^2 + K - \mu/2}\; t),$$

where $c_k, \tilde{c}_k$ are new constants that depend on a_k, b_k, c, k, and μ.

6.1. **Other boundary conditions.** What if we change the boundary conditions to Neumann boundary conditions? Then the wave equation reads

$$(7)\qquad \begin{cases} u_{tt}(t,x) - c^2 u_{xx}(t,x) = 0, & t > 0, x \in (0,1) \\ u_x(t,0) = u_x(t,1) = 0, & t > 0 \\ u(0,x) = u_0(x), & x \in (0,1) \\ u_t(0,x) = g_0(x), & x \in (0,1) \end{cases}.$$

The only thing that changes is that our eigenvalue problem for the Fourier series changes to

$$-X_k(x)'' = \lambda_k X_k(x) \text{ on } (0,1), \quad X_k'(0) = X_k'(1) = 0.$$

We know the solution to this problem are cosines $X_k(x) = \cos(k\pi x)$ with $\lambda_k = k^2\pi^2$ for $k = 0, \ldots$, so the solution to (7) is

$$u(t,x) = \sum_{k=0}^{\infty} T_k(t) \cos(k\pi x)$$

where T_k solves

$$T_k''(t) + (c^2k^2\pi^2)T_k(t) = 0, \quad T_k(0) = \tilde{a}_k, \quad T_k'(0) = \tilde{b}_k.$$

Here, we have written

$$u_0(x) = \sum_{k=0}^{\infty} \tilde{a}_k \cos(k\pi x), \quad g_0(x) = \sum_{k=0}^{\infty} \tilde{b}_k \cos(k\pi x).$$

6.2. **Change in domain.** A last fun example is what happens if we change the domain from $(0,1)$ to $(0,L)$. With Dirichlet BC, we can think of this as holding down the fret of a guitar and then plucking the string. The wave equation here is

$$\begin{cases} u_{tt}(t,x) - c^2 u_{xx}(t,x) = 0, & t > 0, x \in (0,L) \\ u(t,0) = u(t,L) = 0, & t > 0 \\ u(0,x) = u_0(x), & x \in (0,L) \\ u_t(0,x) = g_0(x), & x \in (0,L) \end{cases}. \tag{8}$$

Again the only part of the procedure that changes is the eigenvalue problem, which is now

$$-X_k(x)'' = \lambda_k X_k(x) \text{ on } (0,L), \quad X_k'(0) = X_k'(L) = 0.$$

The solutions to this eigenvalue problem are $X_k(x) = \sin\left(\frac{k\pi}{L}x\right)$ with $\lambda_k = \left(\frac{k\pi}{L}\right)^2$. The solution to (8) is

$$u(t,x) = \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{ck\pi}{L}t\right) + \frac{b_k}{ck\pi} \sin\left(\frac{ck\pi}{L}t\right)\right) \sin(k\pi x)$$

where

$$u_0(x) = \sum_{k=0}^{\infty} a_k \sin\left(\frac{k\pi}{L}x\right), \quad g_0(x) = \sum_{k=0}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right).$$

Remark 6.1 (harmonics of a string)**.** If we were to pluck a string on a guitar of length L, the various frequencies you would here would be $\{\frac{k\pi}{L}\}_{k=1}^{\infty} = \{\sqrt{\lambda_k}\}_{k=1}^{\infty}$. If we press down halfway down a guitar string, then we would here harmonics one octave higher. In general, there is a map φ that maps $(0,L)$ to a sequence of harmonics $\{\sqrt{\lambda_k}\}_{k=1}^{\infty}$. You can see that the map is one to one. That is, if the harmonics we would here are the same, then the interval is the same.

In general, this question was asked for 2D domains and the eigenvalues of the Laplacian. If we solve

$$-u_{xx} - u_{yy} = \lambda u \text{ in } \Omega, \quad u|_{\partial\Omega} = 0,$$

and map Ω to harmonics (essentially λ), is this map 1-1? This question was asked by Mark Kac in "Can One Hear the Shape of a Drum?" in 1966. The answer is no and was answered in 1992.

7. Energy estimates and uniqueness of solutions on an interval

We return to the wave equation on an interval

$$\begin{cases} u_{tt}(t,x) - c^2 u_{xx}(t,x) = f(t,x), & t > 0, x \in (0,1) \\ u(t,0) = u(t,1) = 0, & t > 0 \\ u(0,x) = u_0(x), & x \in (0,1) \\ u_t(0,x) = g_0(x), & x \in (0,1) \end{cases}, \tag{9}$$

and now discuss energy estimates and uniqueness. We first begin with the energy

$$E(t) = \frac{1}{2}\int_0^1 u_t(t,x)^2 + c^2 u_x(t,x)^2 dx. \tag{10}$$

Notice that this an energy in the familiar physics sense. Here u_t^2 is like mv^2 or kinetic energy and $c^2 u_x(t,x)^2$ is like an energy similar to the potential energy of a stretch spring. We know from physics that energy should be conserved unless there is outside work being put into the system. In fact, the wave equation has a conservation of energy. We compute the derivative of the energy and apply Leibniz rule and chain rule:

$$\frac{d}{dt}E(t) = \int_0^1 u_t(t,x)u_{tt}(t,x) + c^2 u_x(t,x)u_{xt}(t,x)dx$$

Notice that we have u_{tt}. In order to use the wave equation, we'd like to have a $-c^2 u_{xx}$ in the integrand. This necessitates integration by parts. If u is smooth, we may swap derivatives $u_{xt}(t,x) = u_{tx}(t,x)$ and then integrate the second term by parts to get

$$\int_0^1 c^2 u_x(t,x)u_{xt}(t,x)dx = -\int_0^1 c^2 u_{xx}(t,x)u_t(t,x)dx + \big(c^2 u_x(t,1)u_t(t,1) - c^2 u_x(t,0)u_t(t,0)\big)$$

Notice that the boundary terms go away because of the boundary condition $u(t,0) = u(t,1) = 0$. Hence,

$$\frac{d}{dt}E(t) = \int_0^1 u_t(t,x)\left(u_{tt}(t,x) - c^2 u_{xx}(t,x)\right)dx = \int_0^1 u_t(t,x)f(t,x)dx.$$

If no outside work is being performed, i.e. $f = 0$, then $\frac{d}{dt}E(t) = 0$ and energy is conserved. Otherwise, we can estimate the energy using standard techniques by first applying Young's inequality

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_0^1 u_t(t,x)f(t,x)dx. \le \frac{1}{2}\int_0^1 u_t(t,x)^2 dx + \frac{1}{2}\int_0^1 f(t,x)^2 dx \\ &\le E(t) + \frac{1}{2}\int_0^1 f(t,x)^2 dx \end{aligned}$$

We can then apply Gronwall's inequality to estimate

$$E(t) \leq e^t \left(E(0) + \frac{1}{2} \int_0^t \int_0^1 f(s,x)^2 dx ds \right),$$

and we can summarize our work in the following proposition.

Proposition 7.1 (energy estimate)**.** Let $u \in C^2$ be a solution to (9). Then the energy in (10) satisfies the estimate

$$E(t) \leq e^t \left(E(0) + \frac{1}{2} \int_0^t \int_0^1 f(s,x)^2 dx ds \right).$$

Moreover, if $f = 0$, we can improve the result to

$$E(t) = E(0).$$

Remark 7.1 (test function)**.** Note that after applying Leibniz rule and integration by parts, we had

$$\frac{d}{dt} E(t) = \int_0^1 u_t(t,x) \left(u_{tt}(t,x) - c^2 u_{xx}(t,x) \right) dx.$$

This tells us that u_t is the function we want to multiply the equation by in order to prove energy estimates for more complicated situations.

An important consequence of the energy estimates and stability is uniqueness, which we have seen many times in this course.

Proposition 7.2 (uniqueness of solution to wave equation on interval)**.** Let $u, v \in C^2$ be solutions to (9). Then $u = v$.

Proof. We set $w = u - v$ and see that w solves

$$\begin{cases} w_{tt}(t,x) - c^2 w_{xx}(t,x) = 0, & t > 0, x \in (0,1) \\ w(t,0) = w(t,1) = 0, & t > 0 \\ u(0,x) = 0, & x \in (0,1) \\ w_t(0,x) = 0, & x \in (0,1) \end{cases}.$$

We then apply the energy estimate to see that $E(t) = E(0) = 0$ for all t. Hence, $w_x(t,x) = 0$ for all t, x. Since we have $w(t,0) = 0$, we can conclude that $w(t,x) = 0$ for all x, t. □

Remark 7.2 (lack of maximum principle)**.** The other concept of stability we have seen in this course in addition to energy estimates is the maximum principle. We note that the wave equation does not have a maximum principle. To see this, consider $u_0(x) = \sin(\pi x), g_0(x) = 0, f(t,x) = 0$. Then the solution to (9) is

$$u(t,x) = \cos(c\pi t)\sin(\pi x),$$

which takes on positive and negative values even though the initial condition is nonnegative. Hence, the wave equation does not have a maximum principle.

8. Numerics for the wave equation

We now begin the discussion of numerics for the wave equation on an interval in (9)

$$\begin{cases} u_{tt}(t,x) - c^2 u_{xx}(t,x) = f(t,x), & t > 0, x \in (0,1) \\ u(t,0) = u(t,1) = 0, & t > 0 \\ u(0,x) = u_0(x), & x \in (0,1) \\ u_t(0,x) = g_0(x), & x \in (0,1) \end{cases} .$$

The first method will consider a second order finite difference for the second derivative:

$$u_{tt}(t_n, x_j) = \frac{u(t_{n+1}, x_j) - 2u(t_n, x_j) + u(t_{n-1}, x_j)}{\tau^2} + \mathcal{O}(\tau^2).$$

Notice that this approximation is at the point (t_n, x_j), so we want to match the approximations for f and u_{xx} also at (t_n, x_j). The resulting iteration for the discrete solution $\mathbf{U}_j^n$ is

$$\frac{\mathbf{U}_j^{n+1} - 2\mathbf{U}_j^n + \mathbf{U}_j^{n-1}}{\tau^2} - c^2 \frac{\mathbf{U}_{j+1}^n - 2\mathbf{U}_j^n + \mathbf{U}_{j-1}^n}{h^2} = \mathbf{f}_j^n. \tag{11}$$

At the boundaries, we use the typical technique and just set

$$\mathbf{U}_0^{n+1} = \mathbf{U}_N^{n+1} = 0.$$

This iteration is has second order truncation error, which can be shown using Taylor expansions at (t_n, x_j).

Proposition 8.1 (consistency and truncation error estimate)**.** Let $\mathbf{u}_j^n = u(t_n, x_j)$ where u solves (9) exactly. Plugging $\mathbf{u}$ into the the above iteration in (11) leads to

$$\frac{\mathbf{u}_j^{n+1} - 2\mathbf{u}_j^n + \mathbf{u}_j^{n-1}}{\tau^2} - c^2 \frac{\mathbf{u}_{j+1}^n - 2\mathbf{u}_j^n + \mathbf{u}_{j-1}^n}{h^2} = \mathbf{f}_j^n + \boldsymbol{\tau}_j^n,$$

where the truncation error satisfies

$$|\boldsymbol{\tau}_j^n| \leq C(\tau^2 |u_{tt}|_{max} + h^2 |u_{xx}|_{max}).$$

Proof. Taylor expansions at (t_n, x_j). □

8.1. **Starting the iteration.** Notice that if we substitute $n = 0$ into the iteration in (11), we get

$$\frac{\mathbf{U}_j^1 - 2\mathbf{U}_j^0 + \mathbf{U}_j^{-1}}{\tau^2} - c^2 \frac{\mathbf{U}_{j+1}^0 - 2\mathbf{U}_j^0 + \mathbf{U}_{j-1}^0}{h^2} = \mathbf{f}_j^0.$$

Notice that we need access to $\mathbf{U}^{-1}$. This is not part of the computational grid. We need to resort to a different method to kick start the iteration that is second order. The idea is to use a Taylor expansion of the exact solution:

$$u(\tau, x_j) = u(0, x_j) + \tau u_t(0, x_j) + \frac{\tau^2}{2} u_{tt}(0, x_j)$$

We can then substitute the initial conditions to get

$$u(\tau, x_j) = u_0(x_j) + \tau g_0(x_j) + \frac{\tau^2}{2} u_{tt}(0, x_j).$$

For the last term, we use the PDE to write $u_{tt}(0, x_j) = f(0, x_j) + c^2 u_{xx}(0, x_j) = f(0, x_j) + c^2 \partial_x^2 u_0(x_j)$, and

$$u(\tau, x_j) = u_0(x_j) + \tau g_0(x_j) + \frac{\tau^2}{2} \left(f(0, x_j) + c^2 \partial_x^2 u_0(x_j) \right).$$

At the discrete level, the above equation becomes

$$\mathbf{U}_j^1 = \mathbf{U}_j^0 + \tau \mathbf{g}_j + \frac{\tau^2}{2} \left(\mathbf{f}_j^0 + c^2 \left(\frac{\mathbf{U}_{j+1}^n - 2\mathbf{U}_j^n + \mathbf{U}_{j-1}^n}{h^2} \right) \right).$$

8.2. **Vector notation.** We can now write the whole scheme using vector notation. Recall the matrix $\mathbf{A}^h$ that is defined by computing the negative second finite difference

$$(\mathbf{A}^h \mathbf{v})_j = -\frac{\mathbf{v}_{j+1} - 2\mathbf{v}_j + \mathbf{v}_{j-1}}{h^2}.$$

We used this matrix in the heat equation. The finite difference scheme we derived has two components.

- **Initialization**: We set

$$\mathbf{U}_j^0 = u_0(x_j), \quad \mathbf{U}^1 = \mathbf{U}^0 + \tau \mathbf{g} + \frac{\tau^2}{2} \left(\mathbf{f}^0 - c^2 \mathbf{A}^h \mathbf{U}^0 \right)$$

- **Iteration**: To compute $\mathbf{U}^{n+1}$ for $n > 0$, we set

$$\frac{\mathbf{U}^{n+1} - 2\mathbf{U}^n + \mathbf{U}^{n-1}}{\tau^2} + c^2 \mathbf{A}^h \mathbf{U}^n = \mathbf{f}^n.$$

8.3. **Stability of the method.** We know the scheme in (11) is consistent. The other ingredient to prove convergence is stability. We first the question of stability in terms of a von Neumann analysis.

To do a von Neumann analysis for a scheme with multiple steps, we think of the application of one step of the scheme as the application of an operator B. That is $\mathbf{U}^n = B\mathbf{U}^{n-1}$ and $\mathbf{U}^{n+1} = B\mathbf{U}^n = B^2\mathbf{U}^{n-1}$. To compute the symbol of B, we replace $\mathbf{U}_j^{n-1} = \mathbf{v}_j = e^{i\xi hj}$, $\mathbf{U}_j^n = S(\xi h)\mathbf{v}_j$, and $\mathbf{U}_j^{n+1} = S(\xi h)^2\mathbf{v}_j$. We then use the iteration in (11):

$$\frac{\mathbf{U}_j^{n+1} - 2\mathbf{U}_j^n + \mathbf{U}_j^{n-1}}{\tau^2} - c^2\frac{\mathbf{U}_{j+1}^n - 2\mathbf{U}_j^n + \mathbf{U}_{j-1}^n}{h^2}$$

and do the substitutions

$$\frac{S(\xi h)^2\mathbf{v}_j - 2S(\xi h)\mathbf{v}_j + \mathbf{v}_j}{\tau^2} - c^2 S(\xi h)\frac{\mathbf{v}_{j+1} - 2\mathbf{v}_j + \mathbf{v}_{j-1}}{h^2} = 0$$

We now multiply by τ^2

$$S(\xi h)^2\mathbf{v}_j - 2S(\xi h)\mathbf{v}_j + \mathbf{v}_j - \frac{c^2\tau^2}{h^2}S(\xi h)\left(\mathbf{v}_{j+1} - 2\mathbf{v}_j + \mathbf{v}_{j-1}\right) = 0$$

Recall that

$$\mathbf{v}_{j+1} - 2\mathbf{v}_j + \mathbf{v}_{j-1} = \mathbf{v}_j\left(e^{i\xi h} + e^{-i\xi h} - 2\right) = \mathbf{v}_j 2\left(\cos(\xi h) - 1\right)$$

Grouping like terms and dividing out the $\mathbf{v}_j$ we are left with

$$S(\xi h)^2 + 2(\lambda^2(1 - \cos(\xi h)) - 1)S(\xi h) + 1 = 0,$$

where $\lambda = \frac{c\tau}{h}$. Since $S(\xi h)$ solves the above quadratic equation, we use the quadratic formula to write

$$S(\xi h) = \frac{-b}{2} \pm \frac{\sqrt{b^2 - 4}}{2}.$$

In order to show the symbol satisfies $|S(\xi h)| \leq 1$, we first need to determine whether the symbol has an imaginary part. From now on, we assume we satisfy the CFL condition

$$\lambda \leq 1 \text{ or equivalently } \tau \leq \frac{h}{c}.$$

We now build up inequalities on b.

$$\begin{aligned}
-1 \leq& -\cos(\xi h) \leq 1 \\
0 \leq& 1 - \cos(\xi h) \leq 2 \\
0 \leq& \lambda^2(1-\cos(\xi h)) \leq 2\lambda^2 \\
-1 \leq& \lambda^2(1-\cos(\xi h)) - 1 \leq 2\lambda^2 - 1 \\
-2 \leq& 2\left[\lambda^2(1-\cos(\xi h)) - 1\right] \leq 4\lambda^2 - 2
\end{aligned}$$

We then use the fact that $\lambda \leq 1$ to conclude $-2 \leq b \leq 2$. Hence, $b^2 - 4 \leq 0$, and

$$S(\xi h) = \frac{-b}{2} + i\frac{\sqrt{4-b^2}}{2}.$$

We now compute the modulus squared of the symbol to get

$$|S(\xi h)|^2 = \frac{b^2}{4} + \frac{4-b^2}{4} = 1.$$

We summarize our work in the following proposition,

Proposition 8.2 (stability of explicit method)**.** Suppose $\tau \leq \frac{h}{c}$. The iteration in (11) is stable in the sense that the symbol satisfies $|S(\xi h)| \leq 1$.

8.4. **von Neumann analysis on interval.** Recall that the von Neumann analysis looked at an infinite grid. In fact, we can repeat the same analysis on a grid of an interval. All we need are the eigenvectors of $\mathbf{A}^h$. Recall we had the following lemma in order to prove a discrete Poincare inequality.

Lemma 8.1 (eigenvalues of $\mathbf{A}^h$)**.** The eigenvalues of $\mathbf{A}^h \in \mathbb{R}^{(N-1)\times(N-1)}$ are

$$\lambda_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right), \quad k = 1,\dots,N-1$$

with eigenvectors

$$\mathbf{v}_i^k = \sin(k\pi x_i).$$

Proof. We just need to verify

$$\left(-\mathbf{v}_{i-1}^k + 2\mathbf{v}_i^k - \mathbf{v}_{i+1}^k\right) = 4\sin^2\left(\frac{k\pi h}{2}\right)\mathbf{v}_i^k.$$

We first split the LHS into two parts

$$-\mathbf{v}_{i-1}^k + 2\mathbf{v}_i^k - \mathbf{v}_{i+1}^k = (\mathbf{v}_i^k - \mathbf{v}_{i-1}^k) + (\mathbf{v}_i^k - \mathbf{v}_{i+1}^k).$$

The first term is

$$\mathbf{v}_i^k - \mathbf{v}_{i-1}^k = \sin(k\pi x_i) - \sin(k\pi x_{i-1}) = \sin(k\pi x_i) - \sin(k\pi(x_i - h)).$$

We then use a angle summation formula for sine:

$$\sin(k\pi(x_i - h)) = \sin(k\pi x_i)\cos(k\pi h) - \cos(k\pi x_i)\sin(k\pi h),$$

so

$$\mathbf{v}_i^k - \mathbf{v}_{i-1}^k = \sin(k\pi x_i)(1 - \cos(k\pi h)) + \cos(k\pi x_i)\sin(k\pi h).$$

We can apply the same techniques to the second term

$$\mathbf{v}_i^k - \mathbf{v}_{i+1}^k = \sin(k\pi x_i)(1 - \cos(k\pi h)) - \cos(k\pi x_i)\sin(k\pi h).$$

Adding both terms together leads to

$$-\mathbf{v}_{i-1}^k + 2\mathbf{v}_i^k - \mathbf{v}_{i+1}^k = 2(1 - \cos(k\pi h))\sin(k\pi x_i) = 2(1 - \cos(k\pi h))\mathbf{v}_i^k.$$

A useful double angle formula for cosine is

$$2(1 - \cos(k\pi h)) = 4\sin^2\left(\frac{k\pi h}{2}\right)$$

□

Once we have the eigenvectors and eigenvalues, we can repeat the von Neumann analysis but use the eigenvalues of $\mathbf{A}^h$. We write the wave equation iteration in matrix notation:

$$\frac{\mathbf{U}^{n+1} - 2\mathbf{U}^n + \mathbf{U}^{n-1}}{\tau^2} + c^2\mathbf{A}^h\mathbf{U}^n = 0$$

and rearrange

$$\mathbf{U}^{n+1} = (2\mathbf{I} - c^2\tau^2\mathbf{A}^h)\mathbf{U}^n - \mathbf{U}^{n-1}$$

We now express

$$\mathbf{U}^{n-1} = \sum_{k=1}^{N-1} a^k\mathbf{v}^k$$

We now seek solutions of the form $a_k^n\mathbf{v}^k$, where $\mathbf{v}^k$ is a single eigenvector of $\mathbf{A}^h$. The iteration now reads

$$a_k^{n+1}\mathbf{v}^k = (2\mathbf{I} - c^2\tau^2\mathbf{A}^h)a_k^n\mathbf{v}^k - a_k^{n-1}\mathbf{v}^k.$$

Notice that $(2\mathbf{I} - c^2\tau^2\mathbf{A}^h)\mathbf{v}^k = (2 - c^2\tau^2\lambda_k)$, so

$$a_k^{n+1}\mathbf{v}^k = (2 - c^2\tau^2\lambda_k)a_k^n\mathbf{v}^k - a_k^{n-1}\mathbf{v}^k.$$

Hence, we have an iteration for the coefficient:

$$a_k^{n+1} = (2 - c^2\tau^2\lambda_k)a_k^n - a_k^{n-1}.$$

The dynamics of this iteration are determined by the roots of the characteristic polynomial

$$s^2 - (2 - c^2\tau^2\lambda_k)s + 1 = 0$$

In order for the iteration to be stable, we want the roots to have complex modulus less than or equal to 1. We again solve for s using the quadratic formula

$$s = \frac{-b}{2} \pm \frac{\sqrt{b^2-4}}{2}$$

where $b = (2 - c^2\tau^2\lambda_k)$. Notice $0 \leq \lambda_k \leq \frac{4}{h^2}$, so

$$2 - \frac{4c^2\tau^2}{h^2} \leq b \leq 2.$$

If we enforce the CFL condition, $c\tau \leq h$, then $-2 \leq b \leq 2$. We then repeat the arguments of the previous section to get that $|s| \leq 1$.

Remark 8.1 (von Neumann analysis)**.** You'll notice that the analysis on the finite grid was essentially the same as the von Neumann analysis that we have studied so far. To adapt the von Neumann analysis to a finite grid in general, one needs to find the eigenvectors and eigenvalues of the relevant discrete operator matrices and then repeat the arguments above. Often the result agrees with the von Neumann analysis on an infinite grid.

8.5. **Energy conserving method for wave equation.** The last section we look at is a derivation of a method that preserves energy. We first look at the continuous problem to draw some inspiration.

Recall that a solution to the wave equation also solved the following system of transport equations

$$\begin{cases} u_t + cu_x = w \\ w_t - cw_x = f \end{cases},$$

which showed us that we'd expect an explicit method to satisfy a CFL condition $\tau \leq h/c$.

The next method will solve the wave equation by looking at the following system of equations

$$\begin{cases} u_t = v \\ v_t - c^2 u_{xx} = f \end{cases}, \tag{12}$$

which has the structure of a heat equation, which is familiar to us. For the rest of the discussion, we'll take $f = 0$.

We shall first rederive conservation of energy from the system in (12). We multiply the second equation by v and integrate over 0 to 1 to get

$$\int_0^1 v_t(t,x)v(t,x) - c^2 u_{xx}(t,x)v(t,x)dx = 0.$$

Recall that we have the standard identity

$$\int_0^1 v_t(t,x)v(t,x)dx = \frac{d}{dt}\frac{1}{2}\int_0^1 v(t,x)^2dx = \frac{d}{dt}\frac{1}{2}\int_0^1 u_t(t,x)^2dx,$$

which is the first part of the energy for the wave equation. Notice that the leftover term we have is

$$-\int_0^1 c^2u_{xx}(t,x)v(t,x)dx,$$

which we would like to rewrite in terms of the second portion of the energy $\int_0^1 c^2|u_x|^2$. In order to move one derivative from u, we integrate by parts and use the homogenous Dirichlet boundary conditions:

$$\begin{aligned}-\int_0^1 c^2u_{xx}(t,x)v(t,x)dx &= \int_0^1 c^2u_x(t,x)v_x(t,x)dx - \cancel{c^2u_x(t,1)v(t,1) + c^2u_x(t,0)v(t,0)} \\ &= \int_0^1 c^2u_x(t,x)v_x(t,x)dx\end{aligned}$$

Notice that the first equation in (12) tells us $v_x(t,x) = u_{tx}(t,x) = u_{xt}(t,x)$, so

$$\int_0^1 c^2u_x(t,x)v_x(t,x)dx = \int_0^1 c^2u_x(t,x)u_{xt}(t,x)dx = \frac{d}{dt}\frac{1}{2}\int_0^1 c^2u_x(t,x)^2dx.$$

We then have

$$\frac{d}{dt}\left[\frac{1}{2}\int_0^1 u_t(t,x)^2dx + \frac{1}{2}\int_0^1 c^2u_x(t,x)^2dx\right] = 0,$$

which is the desired conservation of energy.

8.5.1. *Derivation of method.* The reason we recapped the derivation of conservation of energy is that arguments in the continuous problem serve as a guide for the design of a numerical method.

To achieve the goal of energy conservation we recall that Crank-Nicholson conserved the energy law exactly for the heat equation. As a result, we'll use Crank-Nicholson. To first discretize the second equation at the time point $t_{n+1/2} = \frac{t_{n+1}+t_n}{2}$:

$$\frac{\mathbf{V}^{n+1} - \mathbf{V}^n}{\tau} + c^2\mathbf{A}^h\left(\frac{\mathbf{U}^{n+1} + \mathbf{U}^n}{2}\right) = 0$$

where $\mathbf{V}^n$ is the variable that is a stand in for an approximation of u_t, we'll decide soon what $\mathbf{V}^n$ should be. Mimicking the proof of energy conservation from the continuous problem and recalling the quadratic identity $(a-b)(a+b) = a^2 - b^2$, we take the dot product of the above equation by $h\frac{\mathbf{V}^{n+1}+\mathbf{V}^n}{2}$ to get

$$h\frac{\mathbf{V}^{n+1}-\mathbf{V}^{n-1}}{\tau}\cdot\frac{\mathbf{V}^{n+1}+\mathbf{V}^n}{2} + hc^2\frac{\mathbf{V}^{n+1}+\mathbf{V}^n}{2}\cdot\mathbf{A}^h\left(\frac{\mathbf{U}^{n+1}+\mathbf{U}^n}{2}\right) = 0.$$

We then use the quadratic identity $(a-b)(a+b) = a^2 - b^2$ to simplify the first term

$$h\frac{\mathbf{V}^{n+1}-\mathbf{V}^{n}}{\tau}\cdot\frac{\mathbf{V}^{n+1}+\mathbf{V}^n}{2} = \frac{1}{2\tau}\left(\|\mathbf{V}^{n+1}\|_{2,h}^2 - \|\mathbf{V}^n\|_{2,h}^2\right).$$

We must now make a decision of how to discretize the first equation $u_t = v$. We have left $\mathbf{V}^n$ undecided so far. In order to follow the proof of the continuous problem, we want the discrete u_t to match what is multiplying the discrete u_{xx} (or $\mathbf{A}^h\frac{\mathbf{U}^{n+1}+\mathbf{U}^n}{2}$) in the equation for $\mathbf{V}$. Hence, we discretize $u_t = v$ with again a Crank-Nicholson type approximation.

$$\frac{\mathbf{U}^{n+1}-\mathbf{U}^n}{\tau} = \frac{\mathbf{V}^{n+1}+\mathbf{V}^n}{2}.$$

Hence, the second term now simplifies using again the quadratic identity

$$\begin{aligned} hc^2\left(\frac{\mathbf{V}^{n+1}+\mathbf{V}^n}{2}\right)\cdot\mathbf{A}^h\left(\frac{\mathbf{U}^{n+1}+\mathbf{U}^n}{2}\right) &= hc^2\left(\frac{\mathbf{U}^{n+1}-\mathbf{U}^n}{\tau}\right)\cdot\mathbf{A}^h\left(\frac{\mathbf{U}^{n+1}+\mathbf{U}^n}{2}\right) \\ &= \frac{c^2}{2\tau}\left(\|\mathbf{U}^{n+1}\|_{\mathbf{A}^h}^2 - \|\mathbf{U}^n\|_{\mathbf{A}^h}^2\right) \end{aligned}$$

To summarize, the method we have is the system

$$\begin{cases} \frac{\mathbf{U}^{n+1}-\mathbf{U}^n}{\tau} = \frac{\mathbf{V}^{n+1}+\mathbf{V}^n}{2} \\ \frac{\mathbf{V}^{n+1}-\mathbf{V}^n}{\tau} + c^2\mathbf{A}^h\left(\frac{\mathbf{U}^{n+1}+\mathbf{U}^n}{2}\right) = 0 \end{cases},$$

which satisfies the following discrete conservation of energy relation:

$$\|\mathbf{V}^{n+1}\|_{2,h}^2 + c^2\|\mathbf{U}^{n+1}\|_{\mathbf{A}^h}^2 = \|\mathbf{V}^n\|_{2,h}^2 + c^2\|\mathbf{U}^n\|_{\mathbf{A}^h}^2.$$

Recall that we have that

$$\|\mathbf{V}^{n+1}\|_{2,h}^2 \approx \int_0^1 u_t(t_{n+1},x)^2dx, \quad c^2\|\mathbf{U}^{n+1}\|_{2,h}^2 \approx c^2\int_0^1 u_x(t_{n+1},x)^2dx,$$

so the discrete conservation of energy law above is precisely a discrete analog of the conservation of energy for the wave equation.

To implement the method, one can implement the system above, or one can simplify the method in terms of just $\mathbf{U}^n$. We begin by writing the second equation of the scheme

$$\frac{\mathbf{V}^{n+1}-\mathbf{V}^{n}}{\tau}+c^2\mathbf{A}^h\left(\frac{\mathbf{U}^{n+1}+\mathbf{U}^{n}}{2}\right)=0$$
$$\frac{\mathbf{V}^{n}-\mathbf{V}^{n-1}}{\tau}+c^2\mathbf{A}^h\left(\frac{\mathbf{U}^{n}+\mathbf{U}^{n-1}}{2}\right)=0.$$

Our next goal is to somehow write the scheme in terms of averages $\frac{\mathbf{V}^{n+1}+\mathbf{V}^{n}}{2}$ in order to use the first equation of the scheme and write everything in terms of $\mathbf{U}^n$. We do this by adding both equations above and dividing by 2:

$$\frac{\mathbf{V}^{n+1}-\mathbf{V}^{n}}{2\tau}+\frac{\mathbf{V}^{n}-\mathbf{V}^{n-1}}{2\tau}+c^2\mathbf{A}^h\left(\frac{\mathbf{U}^{n+1}+2\mathbf{U}^{n}+\mathbf{U}^{n-1}}{4}\right)=0$$

To write in terms of $\frac{\mathbf{V}^{n+1}+\mathbf{V}^{n}}{2}$, we regroup the first two terms:

$$\frac{\mathbf{V}^{n+1}+\mathbf{V}^{n}}{2\tau}-\frac{\mathbf{V}^{n}+\mathbf{V}^{n-1}}{2\tau}+c^2\mathbf{A}^h\left(\frac{\mathbf{U}^{n+1}+2\mathbf{U}^{n}+\mathbf{U}^{n-1}}{4}\right)=0$$

Notice that

$$\frac{\mathbf{V}^{n+1}+\mathbf{V}^{n}}{2\tau}=\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n}}{\tau^2}$$
$$\frac{\mathbf{V}^{n}+\mathbf{V}^{n-1}}{2\tau}=\frac{\mathbf{U}^{n}-\mathbf{U}^{n-1}}{\tau^2}.$$

Hence,

$$\underbrace{\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n}}{\tau^2}-\frac{\mathbf{U}^{n}-\mathbf{U}^{n-1}}{\tau^2}}_{=\frac{\mathbf{U}^{n+1}-2\mathbf{U}^{n}+\mathbf{U}^{n-1}}{\tau^2}}+c^2\mathbf{A}^h\left(\frac{\mathbf{U}^{n+1}+2\mathbf{U}^{n}+\mathbf{U}^{n-1}}{4}\right)=0$$

and we have

$$\frac{\mathbf{U}^{n+1}-2\mathbf{U}^{n}+\mathbf{U}^{n-1}}{\tau^2}+c^2\mathbf{A}^h\left(\frac{\mathbf{U}^{n+1}+2\mathbf{U}^{n}+\mathbf{U}^{n-1}}{4}\right)=0.$$

Notice that the first term is a second finite difference to approximate $u_{tt}(t_n,x)$ and the second term is a time averaged approximation of $-u_{xx}(t_n,x)$, so we have a second order consistent method.

All of our work can now be summarized in the following proposition,

Proposition 8.3 (energy conserving method for the wave equation)**.** Let $\mathbf{U}^n$ be a sequence of grid functions that satisfy the iteration

$$\frac{\mathbf{U}^{n+1} - 2\mathbf{U}^n + \mathbf{U}^{n-1}}{\tau^2} + c^2 \mathbf{A}^h \left(\frac{\mathbf{U}^{n+1} + 2\mathbf{U}^n + \mathbf{U}^{n-1}}{4} \right) = 0.$$

Then, this method is second order consistent with truncation error

$$|\boldsymbol{\tau}_j^n| \leq C \left(\tau^2 |u_{tttt}|_{max} + \tau^2 |u_{xxtt}|_{max} + h^2 \tau^2 |u_{xxxx}|_{max} \right),$$

and the method conserves energy in the sense that

$$\|\mathbf{V}^{n+1}\|_{2,h}^2 + c^2 \|\mathbf{U}^{n+1}\|_{\mathbf{A}^h}^2 = \|\mathbf{V}^n\|_{2,h}^2 + c^2 \|\mathbf{U}^n\|_{\mathbf{A}^h}^2,$$

where $\frac{\mathbf{V}^{n+1} + \mathbf{V}^n}{2} = \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\tau}$.

Proof. Energy conservation is a consequence of the arguments made above. The truncation error follows from standard Taylor arguments centered at (t_n, x_j). The only additional term you have to deal with is to show for any $f \in C^2$:

$$\frac{f(t_{n+1}) + 2f(t_n) + f(t_{n-1})}{4} = f(t_n) + \mathcal{O}(\tau^2).$$

□