COMPUTATIONAL PDE LECTURE 25

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1. Outline of this lecture

- Discuss schemes for nonlinear conservation laws

2. Setup

We are trying to solve the following nonlinear conservation law on the whole real line:

$$\begin{cases} u_t(t,x) + \partial_x[f(u(t,x)] = 0, & t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

An important concept that solutions to (2) is mass conservation

$$\frac{d}{dt}\int_{\mathbb{R}} u(t,x)dx = 0,$$

which we have shown in previous lectures. An important consequence of mass conservation is what is known as order preservation.

Proposition 2.1 (order preservation)**.** Let u, v be two solutions (2) with $u(0,x) = u_0(x)$ and $v(0,x) = v_0(x)$. If $u_0(x) \leq v_0(x)$ for all $x \in \mathbb{R}$, then $u(t,x) \leq v(t,x)$ for all $x \in \mathbb{R}$, $t > 0$.

Proof. A fact that we first use about is that the difference of two solutions is controlled by the difference of the initial conditions in an integral sense, i.e.

$$\int_{\mathbb{R}} |v(t,x) - u(t,x)|dx \leq \int_{\mathbb{R}} |v_0(x) - u_0(x)|dx$$

This fact is actually quite difficult to prove. Once we have it though, we use the assumption that $v_0(x) - u_0(x) \geq 0$ and we can drop the absolute value and apply the mass conservation property

$$\int_{\mathbb{R}} |v_0(x) - u_0(x)|dx = \int_{\mathbb{R}} v_0(t,x) - u_0(t,x)dx = \int_{\mathbb{R}} v(t,x) - u(t,x)dx.$$

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Hence, we have

$$\int_{\mathbb{R}} |v(t,x) - u(t,x)| dx \le \int_{\mathbb{R}} v(t,x) - u(t,x) dx.$$

This implies $v(t,x) \ge u(t,x)$ because if not, then the integral on the LHS would become larger and contradict the above inequality. □

3. Order preserving numerical schemes

A successful numerical method for (2) will mimic the order preservation property. This property is known as monotonicity:

Definition 3.1 (monotonicity)**.** We say a numerical iteration that maps $\mathbf{U}^n$ to $\mathbf{U}^{n+1}$ is monotone if for $\mathbf{V}_j^n \ge \mathbf{U}_j^n$ for all j, then $\mathbf{V}_j^{n+1} \ge \mathbf{U}_j^{n+1}$.

We now list some monotone schemes for (2).

- **Upwind** Suppose $f'(u) \ge 0$ for all $u \in \mathbb{R}$, then the upwind scheme is

$$\frac{\mathbf{U}_j^{n+1} - \mathbf{U}_j^n}{\tau} + \frac{f(\mathbf{U}_j^n) - f(\mathbf{U}_{j-1}^n)}{h} = 0$$

- **Lax-Fredrichs** The Lax-Fredrichs scheme is

$$\frac{\mathbf{U}_j^{n+1} - \frac{1}{2}(\mathbf{U}_{j+1}^n + \mathbf{U}_{j-1}^n)}{\tau} + \frac{f(\mathbf{U}_{j+1}^n) - f(\mathbf{U}_{j-1}^n)}{2h} = 0$$

These schemes are both monotone under assumptions on f. We first list the result for upwind:

Proposition 3.1 (monotonicity of upwind)**.** Suppose $f'(u) \ge 0$ for all $u\mathbb{R}$ and $f''(u) \ge 0$ for all $u \in \mathbb{R}$, i.e. f is convex. Then upwind is monotone as long as the CFL condition $\tau \le \frac{h}{|f'(u_0)|_{max}}$.

Proof. We only proof the result for $f(u) = cu$ for $c > 0$, which is the transport equation. Assume $\mathbf{V}_j^n \ge \mathbf{U}_j^n$ for all j. We now write

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n + \frac{\tau c}{h}(\mathbf{U}_{j-1}^n - \mathbf{U}_j^n)$$
$$\mathbf{V}_j^{n+1} = \mathbf{V}_j^n + \frac{\tau c}{h}(\mathbf{V}_{j-1}^n - \mathbf{V}_j^n)$$

We now subtract to get

$$\mathbf{V}_j^{n+1} - \mathbf{U}_j^{n+1} = \left(1 - \frac{\tau c}{h}\right)\left(\mathbf{V}_j^n - \mathbf{U}_j^n\right) + \frac{\tau c}{h}\left(\mathbf{V}_{j-1}^n - \mathbf{U}_{j-1}^n\right).$$

Notice that by the CFL condition, $1 - \frac{\tau c}{h} \ge 0$, and by the fact that we used upwinding, we have $\frac{\tau c}{h} \ge 0$. Since we also have $\mathbf{V}_j^n - \mathbf{U}_j^n \ge 0$ for all j, then $\mathbf{V}_j^{n+1} \ge \mathbf{U}_j^{n+1}$ for all j, which is the desired result □

Remark 3.1. We note that the proof of monotonicity for the transport equation is essentially the proof of ∞ norm stability of upwind. In fact, monotonicity implies ∞ norm stability.

Remark 3.2. To prove the above monotonicity result for generic f that is convex, you want to repeat the same arguments but use properties of convex functions like

$$f(y) \geq f(x) + f'(x)(y - x),$$

which means the tangent line of a convex function lies below the graph of said function.

We also have that Lax-Fredrichs is monotone, though we will will not prove it.

Proposition 3.2 (monotonicity of Lax-Fredrichs)**.** Suppose $f''(u) \geq 0$ for all $u \in \mathbb{R}$, i.e. f is convex. Then Lax-Fredrichs is monotone as long as the CFL condition $\tau \leq \frac{h}{|f'(u_0)|_{max}}$.

Remark 3.3 (Lax-Wendroff is not monotone)**.** The Lax-Wendroff scheme we developed in previous lectures is not monotone for the transport equation. This can be seen by looking at the following initial condition from Recitation

$$u_0(x) = \begin{cases} 1, & x \in (1/4, 5/4), \\ 0, & \text{otherwise} \end{cases}.$$

We can see numerically in the next plot that Lax-Wendroff is not monotone, while upwind and Lax-Fredrichs are monotone.