COMPUTATIONAL PDE LECTURE 24

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1. Outline of this lecture

- Discuss Lax-Fredrich's scheme and derive the Lax-Wendroff scheme

2. Setup

We are trying to solve the transport equation on the whole real line:

$$\begin{cases} u_t(t,x) + cu_x(t,x) = f(t,x), \quad t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

3. Other methods for transport

Recall that upwind is stable, but lower order. We now discuss some strategies to get higher order methods.

3.1. **Centered Difference is unstable.** A natural next scheme to achieve higher accuracy in space would be to replace the upwind term with a centered finite difference. The resulting iteration is

$$D_\tau \mathbf{U}_j^n + c\left(\frac{\mathbf{U}_{j+1}^{n-1} - \mathbf{U}_{j-1}^{n-1}}{2h}\right) = \mathbf{f}_j^{n-1}. \tag{2}$$

However, this scheme is always unstable! We proved this using von Neumann Stability analysis

3.2. **Another scheme: Lax-Frederich's Scheme.** In order to save the centered difference, we now look at a way to "add diffusion" or "smooth out" the numerical solution. One way to smooth out is to either add a diffusion term directly or by taking an average. The Frederich's scheme works by taking an average :

$$\frac{1}{\tau}\left[\mathbf{U}_j^n - \left(\frac{\mathbf{U}_{j-1}^{n-1} + \mathbf{U}_{j+1}^{n-1}}{2}\right)\right] + c\left(\frac{\mathbf{U}_{j+1}^{n-1} - \mathbf{U}_{j-1}^{n-1}}{2h}\right) = \mathbf{f}_j^{n-1}. \tag{3}$$

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One can show that this scheme is stable and is a second order scheme (in h for fixed τ) for the following modified PDE:

$$u_t + cu_x - \frac{h^2}{2\tau}u_{xx} = 0.$$

However, note that we want to take $\tau \leq \frac{h}{c}$, so if $\tau = \frac{h}{c}$ and $h \to 0$, we no longer have a second order scheme due to the $\frac{h^2}{2\tau} \approx h$ numerical diffusion term.

We now state the specific results for the Lax-Frederich's scheme in (3) but leave the proofs as exercises.

Proposition 3.1 (stability of Lax-Frederich's)**.** Let $\mathbf{U}^n$ solve the Frederich's iteration in (3). Assume the CFL condition $\tau \leq \frac{h}{c}$. Then,

$$\|\mathbf{U}^n\|_\infty \leq \|\mathbf{U}^{n-1}\|_\infty + \tau\|\mathbf{f}^{n-1}\|_\infty.$$

Proposition 3.2 (consistency of Lax-Frederich's)**.** Let $\mathbf{u}_j^n = u(t_n, x_j)$ be the exact solution of the transport equation. Then,

$$\frac{1}{\tau}\left[\mathbf{u}_j^n - \left(\frac{\mathbf{u}_{j-1}^{n-1} + \mathbf{u}_{j+1}^{n-1}}{2}\right)\right] + c\left(\frac{\mathbf{u}_{j+1}^n - \mathbf{u}_{j-1}^n}{2h}\right) = \mathbf{f}_j^{n-1} + \boldsymbol{\tau}_j^{n-1}.$$

where there is a $C > 0$ independent of h, τ such that

$$|\boldsymbol{\tau}_j^{n-1}| \leq C\left(h^2|u_{xxx}|_{max} + \tau|u_{tt}|_{max} + \frac{h^2}{\tau}|u_{xx}|_{max}\right)$$

Proposition 3.3 (error estimate of Lax-Frederich's)**.** Let $\mathbf{U}^n$ solve the Frederich's iteration in (3). Assume the CFL condition $\tau \leq \frac{h}{c}$. Let $\mathbf{u}_j^n = u(t_n, x_j)$ be the exact solution of the transport equation. Then, the error satisfies

$$\|\mathbf{u}^n - \mathbf{U}^n\|_\infty \leq Ct_n\left(h^2|u_{xxx}|_{max} + \tau|u_{tt}|_{max} + \frac{h^2}{\tau}|u_{xx}|_{max}\right).$$

3.3. **Second order method: Lax-Wendroff.** We now derive a second order scheme using the symbol of an operator. Suppose $u(t, x) = e^{i\xi x}$. Then if u solves the transport equation, we have

$$u(t + \tau, x) = e^{i\xi(x - c\tau)} = e^{-i\xi c\tau}u(t, x).$$

In some sense, the true symbol we'd like to approximate is $S(\xi) = e^{-i\xi c\tau}$. If we are able to approximate the true symbol at $\xi = 0$ for ξ small, then we are likely to have a convergent method. One way to do this would be to write a Taylor expansion of the symbol

$$S(\xi) = 1 - c\tau i\xi - \frac{c^2\tau^2}{2}\xi^2 + \mathcal{O}(\xi^3).$$

This can be summarized in the following theorem that we will not prove.

Theorem 3.1. We define an explicit finite difference method by the following iteration:

$$\mathbf{U}_j^{n+1} = \sum_{k=-\infty}^{\infty} a_k \mathbf{U}_{j+k}^n.$$

Let $S(\xi h)$ be the symbol of the above iteration. That is, if $\mathbf{v}_j = e^{i\xi hj}$, then

$$S(\xi h)\mathbf{v}_j = \sum_{k=-\infty}^{\infty} a_k \mathbf{v}_{j+k}.$$

Further assume that for for $\tau \leq \frac{h}{c}$, we have $|S(\xi h)| \leq 1$ for all ξ and there is a $\xi^* > 0$ such that if $|\xi h| < \xi^*$, we have

$$|S(\xi h) - e^{-i\xi c\tau}| \leq C|\xi h|^{r+1}.$$

Then the finite difference method satisfies the following error estimate for $\tau \leq \frac{h}{c}$:

$$\|\mathbf{u}^n - \mathbf{U}^n\|_{2,h} \leq \mathcal{O}(t_n(\tau^r + h^r)).$$

3.3.1. *Derivation of Lax-Wendroff.* We won't prove this theorem, but will use it to derive a second order scheme. In order to achieve $|S(\xi) - e^{-i\xi c\tau}| \leq C|\xi|^3$, we need to match the Taylor expansion of the true symbol. We define our scheme with three coefficients

$$\mathbf{U}_j^{n+1} = a_{-1}\mathbf{U}_{j-1}^n + a_0\mathbf{U}_j^n + a_1\mathbf{U}_{j+1}^n,$$

and now compute the symbol as

$$S(\xi h) = (a_{-1}e^{-i\xi h} + a_0 + a_1 e^{i\xi h}),$$

with

$$S'(\xi h) = (-iha_{-1}e^{-i\xi h} + iha_1 e^{i\xi h})$$

and

$$S''(\xi h) = (-h^2 a_{-1}e^{-i\xi h} - h^2 a_1 e^{i\xi h})$$

We now want

$$S(0) = 1, \quad S'(0) = -c\tau i, \quad S''(0) = -c^2\tau^2$$

Setting $S(0) = 1$ leads to $a_{-1} + a_0 + a_1 = 0$. We compute

$$S'(0) = -iha_{-1} + a_1 hi.$$

Hence, we want $-\frac{c\tau}{h} = a_1 - a_{-1}$. Finally, we compute

$$S''(0) = -h^2 a_{-1} - h^2 a_1,$$

and require $a_{-1} + a_1 = \frac{c^2\tau^2}{2h^2}$. We are left with the linear system with $\lambda = \tau/h$:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{-1} \\ a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -c\lambda \\ c^2\lambda^2 \end{pmatrix}$$

whose solution is

$$a_{-1} = \frac{1}{2}(c^2\lambda^2 + c\lambda), \quad a_0 = 1 - c^2\lambda^2, \quad a_1 = \frac{1}{2}(c^2\lambda^2 - c\lambda).$$

We now have a second order scheme, which is the Lax-Wendroff scheme:

$$\mathbf{U}_j^{n+1} = \frac{1}{2}(c^2\lambda^2 + c\lambda)\mathbf{U}_{j-1}^n + (1 - c^2\lambda^2)\mathbf{U}_j^n + \frac{1}{2}(c^2\lambda^2 - c\lambda)\mathbf{U}_{j+1}^n$$