COMPUTATIONAL PDE LECTURE 1

LUCAS BOUCK

1. Learning goals of the course

- Various types of partial differential equations (PDEs)
  - Second order elliptic (applications in electrostatics, steady state heat distribution)
  - Second order parabolic (applications in heat transfer, diffusion, fluids)
  - First order hyperbolic (applications in transport, fluids, models of car traffic)
  - Second order hyperbolic (applications in fluids, electrodynamics)
- Some analytical techniques to find solutions of PDEs. Some examples of techniques include
  - Separation of variables
  - Green's Functions
  - Method of characteristics
- Qualitative behavior of solutions, which may include
  - Maximum and variational principles for elliptic problems
  - Energy estimates for parabolic equations
  - Conservation of energy for hyperbolic equations
- Numerical techniques (specifically finite differences) to approximate the solution of a PDE leveraging computational power. Other topics around finite differences will include:
  - Discrete maximum principle for elliptic equations
  - Euler and Crank-Nicolson time stepping for parabolic equations
  - von Neumann analysis for finite difference schemes for parabolic equations
  - Upwind finite differences for first order hyperbolic equations

2. Topics from Ordinary Differential Equations (ODEs)

Recall that an ordinary differential equation is about solving for some unknown function y, where y solves some equation.

Definition 2.1 (ordinary differential equation)**.** *An ordinary differential equation is an equation of some unknown function* $y : [0, T] \to \mathbb{R}$ *and its derivatives* $\frac{\mathrm{d}^k \mathbf{y}}{\mathrm{d}t^k}$

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Below are some examples that you may have seen in previous courses.

Example 2.1 (exponential growth/decay). *Let* $c \in \mathbb{R}$. *The equation*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = cy,$$

describes exponential growth for $c > 0$ *and exponential decay for* $c < 0$. *This is an example of linear, first order ODE.*

Example 2.2 (mass-spring system). *Let* $m, k > 0$. *The equation*

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + ky = 0,$$

describes the frictionless movement of an object with mass m *on the end of spring with spring constant* k. *This is an example of a linear, second order ODE.*

Example 2.3 (more general ODE). *Let* $f : [0, T] \times \mathbb{R} \to \mathbb{R}$. *The equation*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y),$$

is describes a general set of first order ODEs. This equation reduces to Example 2.1 if $f(t, y) = cy$.

2.1. **Relevant Questions for ODEs.**

- Does there exist solutions y to the ODE? If there exists a solution, is the solution the only one (i.e. uniqueness)?
  - This question is can be answered quite generally for ODEs. For instance, the initial value problem

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} & = f(y) \\ y(0) & = y_0 \end{cases}$$

    has a unique solution on some interval $(-T, T)$ for T sufficiently small if f is continuously differentiable. This is a consequence of the Picard-Lindelöf Theorem.
- Can we construct analytical solutions to solve the ODE?
  - Often, yes. This was the topic of your previous courses in ODEs and you may have seen separation of variables, Laplace transform, Fourier transform, and some other techniques to solve ODEs. It is also sometimes difficult to solve these equations analytically, and we may need to resort to the next equation
- How can we design numerical methods to approximate the solution of an ODE?

- You may have seen Euler's method for approximating the solution to an ODE. For the initial value problem

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} &= f(t,y) \\ y(0) &= y_0 \end{cases}.$$

We may approximate the solution with a time-step $\tau > 0$ and apply the following iteration

$$\begin{cases} y_\tau^0 &= y_0 \\ y_\tau^{k+1} &= y_\tau^k + \tau f(t^k, y_\tau^k) \end{cases}.$$

The sequence y_τ^k approximates the value of the true solution y at time $t_k = k\tau$. This course on PDEs will cover numerical methods like Euler's method, but for partial differential equations

3. Partial Differential Equations (PDEs)

For the rest of this course, we will use the notation $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ to be an open domain on which we will solve the PDE. We now state the definition of a PDE.

Definition 3.1 (partial differential equation)**.** *An partial differential equation is an equation of some unknown function* $u : \Omega \to \mathbb{R}$ *and its partial derivatives* u_x, u_{xx}, u_y, u_{xy}*, etc.*

Remark 3.1 (partial derivative notation)**.** *We will denote* u_x *to denote the partial derivative of* u *with respect to the variable* x*, i.e.* $\frac{\partial u}{\partial x}$*. Some other examples that we might see are*

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad u_t = \frac{\partial u}{\partial t}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}$$

3.1. **Relevant Questions for PDEs.** Below are the questions that we will touch on for PDEs. One key difference between ODEs and PDEs is that the answer to these questions are quite general for ODEs, and these answers vary wildly depending on the type of PDE.

- Does there exist solutions u to the PDE? If there exists a solution, is the solution the only one (i.e. uniqueness)?
  - The question of existence can be quite difficult, but we will see that constructing analytical solutions can provide proof of existence of solutions. We will focus more on showing uniqueness, which will prove useful in analyzing the stability of numerical methods. Such topics we will cover on uniqueness are: maximum principle for elliptic problems, energy dissipation for parabolic problems, and energy or mass conservation for hyperbolic problems
- Can we construct analytical solutions to solve the PDE?

  - Sometimes, but it is quite difficult for PDEs. The techniques we will see are Green's functions, separation of variables, and methods of characteristics.
- How can we design numerical methods to approximate the solution of an PDE?
  - This will be the main focus of this course. We will implement **finite difference methods** to approximate the solution of a PDE.

3.2. **Examples.**

Example 3.1 (Poisson's or Laplace's equation)**.** *We consider the set with piecewise smooth boundary. We seek* $u : \mathbb{R}^2 \to \mathbb{R}$ *to be a function that satisfies*

$$u_{xx}(x, y) + u_{yy}(x, y) = f(x, y) \text{ for } (x, y) \in \mathbb{R}^2.$$

This equation is known as Poisson's equation, which is a linear, second order, **elliptic** *equation. A special case is Laplace's equation when* $f(x, y) = 0$ *for all* $(x, y) \in \mathbb{R}^2$*. One such applications for Poisson's equation is electrostatics, where* u *is the electric potential that arises from a charge density* f*.*

Example 3.2 (heat equation)**.** *We consider a set* $(0, t)\times\mathbb{R}$ *the set, which is a linear, second order,* **parabolic** *equation. We seek* $u : (0, t) \times \mathbb{R} \to \mathbb{R}$ *to be a function that satisfies*

$$u_t(t, x) - u_{xx}(t, x) = f(t, x) \text{ for } (t, x) \in (0, t) \times \mathbb{R}.$$

This equation is known as the heat equation. This equation models heat transfer with some heat source/sink f*.*

Example 3.3 (transport equation)**.** *We consider a set* $(0, t) \times \mathbb{R}$*. We seek* $u : (0, t) \times \mathbb{R} \to \mathbb{R}$ *to be a function that satisfies*

$$u_t(t, x) + c(t, x)u_x(t, x) = f(t, x) \text{ for } (t, x) \in (0, t) \times \mathbb{R}.$$

This equation is known as the transport equation, which is a linear, first order, **hyperbolic** *equation.*

Example 3.4 (wave equation)**.** *Let* $c > 0$*. We consider a set* $(0, t) \times \mathbb{R}$*. We seek* $u : (0, t) \times \mathbb{R} \to \mathbb{R}$ *to be a function that satisfies*

$$u_{tt}(t, x) - c^2u_{xx}(t, x) = f(t, x) \text{ for } (t, x) \in (0, t) \times \mathbb{R}.$$

This equation is known as the wave equation, is a linear, second order, **hyperbolic** *equation. This equation often describes the motion of a vibrating string and pops up in electrodynamics.*

3.3. **Elliptic, Parabolic, Hyperbolic Terminology for Second Order Problems.** We now go through a rough procedure for explaining why we denote these equations as elliptic, parabolic, and hyperbolic and then highlight a more mechanical procedure for second order equations with 2 independent variables.

3.3.1. *Heuristic.* The following heuristic procedure hopefully helps explain the different terminology for these equations. The procedure is as follows.

- Take an equation and replace the partial derivatives of u with the independent variables in the subscript. For instance,
  - u_t will be replaced with t
  - u_{xx} would be replaced with $x \cdot x = x^2$
  - u_{xy} would be replaced with xy and so on.
- Take any term not multiplying second order derivatives like $f(x, y)$, and replace them with a constant like 1.
- The algebraic equation built by this procedure will describe an ellipse, parabola, or hyperbola for second order equations.

We now go over the example equations:

- Poisson's equation $u_{xx} + u_{yy} = f$ will become $x^2 + y^2 = 1$, which is the equation for a circle, which is a special case of any ellipse.
- The heat equation $u_t - u_{xx} = f$ would become $t - x^2 = 1$, which is the equation for a parabola.
- The wave equation $u_{tt} - c^2 u_{xx} = f$ would become $t^2 - c^2 x^2 = 1$, which is the equation for a hyperbola as long as $c \neq 0$.

This procedure is just a heuristic. We can make things more concrete.

3.3.2. *Classification of Elliptic, Parabolic, and Hyperbolic 2nd Order Equations in 2D.* Consider the following second order linear PDE

$$A(x,y)u_{xx} + \frac{B(x,y)}{2}u_{xy} + \frac{B(x,y)}{2}u_{yx} + C(x,y)u_{yy} + \mathbf{b}(x,y) \cdot \nabla u(x,y) + c(x,y)u(x,y) = f(x,y),$$

and write the matrix

$$\mathbf{M}(x,y) = \begin{pmatrix} A(x,y) & \frac{B(x,y)}{2} \\ \frac{B(x,y)}{2} & C(x,y) \end{pmatrix}.$$

Note that at any point (x, y), the matrix $\mathbf{M}(x, y)$ is symmetric, so it has 2 real eigenvalues $\lambda_1(x, y)$ and $\lambda_2(x, y)$. The table below gives a classification of the PDE at the point (x, y):

Elliptic	$\lambda_1(x,y) \neq 0$, $\lambda_2(x,y) \neq 0$ and $\lambda_1(x,y), \lambda_2(x,y)$ have the same sign
Parabolic	$\lambda_1(x,y) = 0$ or $\lambda_2(x,y) = 0$
Hyperbolic	$\lambda_1(x,y) \neq 0$, $\lambda_2(x,y) \neq 0$ and $\lambda_1(x,y), \lambda_2(x,y)$ have the opposite sign

We now use this criteria on the 2nd order examples from before.

- Poisson's equation $u_{xx} + u_{yy} = f$:

$$\mathbf{M}(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies \lambda_1 = \lambda_2 = 1,$$

so Poisson's equation is elliptic.

- Heat equation $u_t - u_{xx} = f$:

$$\mathbf{M}(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \implies \lambda_1 = 0, \text{ and } \lambda_2 = 1,$$

so the heat equation is parabolic.

- Wave equation $u_{tt} - c^2 u_{xx} = f$:

$$\mathbf{M}(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -c^2 \end{pmatrix} \implies \lambda_1 = 1, \text{ and } \lambda_2 = -c^2,$$

so the wave equation is hyperbolic as long as $c \neq 0$.

Remark 3.2 (lower order terms and PDE classification)**.** *The lower order terms* $\mathbf{b}(x,y)\cdot\nabla u + c(x,y)u(x,y)$ *did not impact the classification of the PDE. What matters are the 2nd order terms.*

Remark 3.3 (vector and matrix notation)**.** *We'll use bold face with lower case letter to denote vectors. For example, at a point* (x,y)*, we have*

$$\mathbf{b}(x,y) = (b_1(x,y), b_2(x,y))^T \in \mathbb{R}^2.$$

Also, the notation $\cdot$ *denotes the dot product of two vectors, so*

$$\mathbf{a}\cdot\mathbf{b} = a_1 b_1 + a_2 b_2.$$

We'll use bold face with upper case letters to denote a matrix. For example at a point (x,y)*, we have*

$$\mathbf{M}(x,y) \in \mathbb{R}^{2\times 2}.$$